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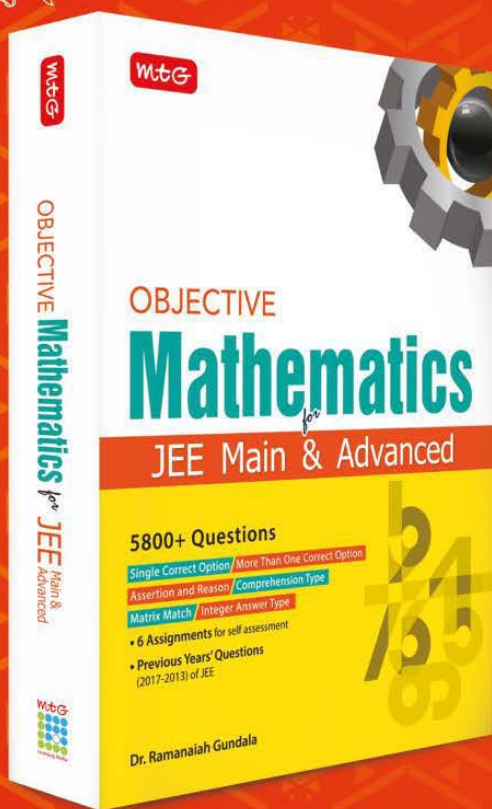
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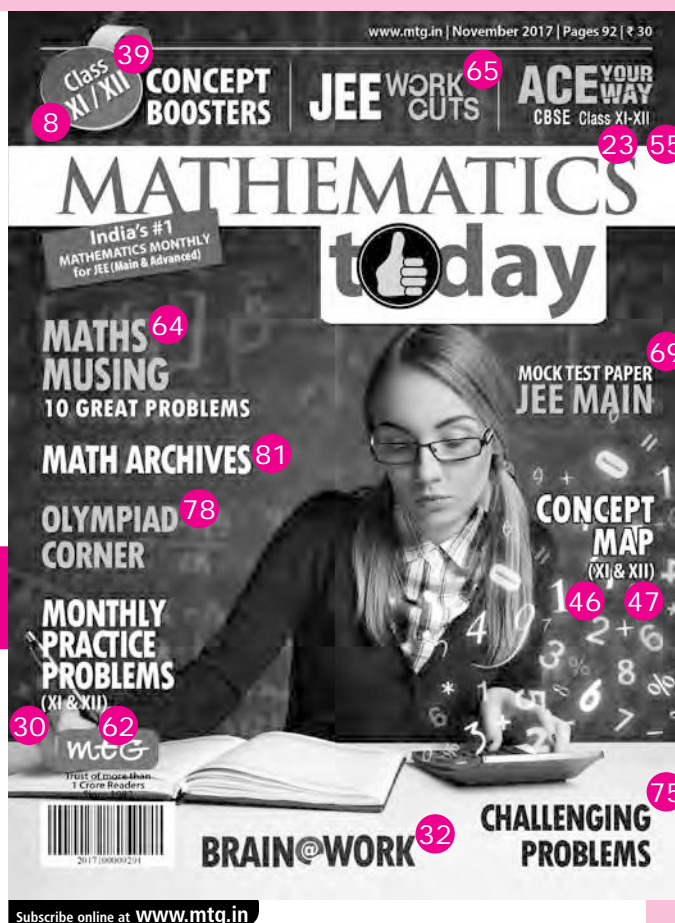
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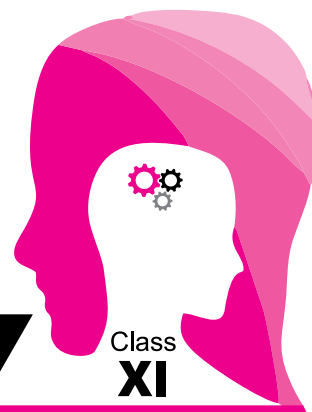
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CONCEPT BOOSTERS

Binomial Theorem and Principle of Mathematical Induction



Class
XI

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

The rule by which any power of binomial can be expanded is called the binomial theorem.

If n is a positive integer and $x, y \in C$ then

$$(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n = \sum_{r=0}^n {}^nC_r \cdot x^{n-r} \cdot y^r$$

Here, ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are called binomial coefficients and ${}^nC_r = \frac{n!}{r!(n-r)!}$ for $0 \leq r \leq n$.

SOME IMPORTANT EXPANSIONS

- $(x-y)^n = {}^nC_0 x^{n-0} y^0 - {}^nC_1 x^{n-1} y^1 + \dots + (-1)^r {}^nC_r x^{n-r} y^r + \dots + (-1)^n {}^nC_n x^0 y^n = \sum_{r=0}^n (-1)^r {}^nC_r x^{n-r} y^r$
- $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n = \sum_{r=0}^n {}^nC_r x^r$
- $(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x^1 + {}^nC_2 x^2 + \dots + (-1)^r {}^nC_r x^r + \dots + (-1)^n {}^nC_n x^n = \sum_{r=0}^n (-1)^r {}^nC_r x^r$
- $(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \dots]$

$$\text{and } (x+y)^n - (x-y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + \dots]$$

- The coefficient of $(r+1)^{\text{th}}$ term in the expansion of $(1+x)^n$ is nC_r .
- The coefficient of x^r in the expansion of $(1+x)^n$ is nC_r .

GENERAL TERM

- The general term of the expansion is $(r+1)^{\text{th}}$ term usually denoted by $T_{r+1} = {}^nC_r x^{n-r} y^r$
- In the binomial expansion of $(x+y)^n$, the p^{th} term from the end is $(n-p+2)^{\text{th}}$ term from beginning.

NUMBER OF TERMS IN THE EXPANSION OF $(a+b+c)^n$ AND $(a+b+c+d)^n$

$$\begin{aligned} (a+b+c)^n &\text{ can be expanded as } \{(a+b)+c\}^n \\ &= (a+b)^n + {}^nC_1 (a+b)^{n-1} (c)^1 + \dots + {}^nC_n c^n \\ &= (n+1) \text{ term} + n \text{ term} + (n-1) \text{ term} + \dots + 1 \text{ term} \\ \therefore \text{ Total number of terms} &= (n+1) + (n) + \dots + 1 \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Similarly, number of terms in the expansion of

$$(a+b+c+d)^n = \frac{(n+1)(n+2)(n+3)}{6}$$

MIDDLE TERM

The middle term depends upon the value of n .

- When n is even**, then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (odd). So there

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is only one middle term i.e., $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term is the middle term.

$$T_{\left[\frac{n}{2} + 1\right]} = {}^nC_{n/2} x^{n/2} y^{n/2}$$

- **When n is odd**, then total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (even). So, there are two middle terms i.e., $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ are two middle terms.

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \quad \text{and}$$

$$T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$$

- When there are two middle terms in the expansion then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest binomial coefficient.

GREATEST TERM AND GREATEST COEFFICIENT

- **Greatest term** : If T_r and T_{r+1} be the r^{th} and $(r + 1)^{\text{th}}$ terms in the expansion of $(1 + x)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1} x^{r-1}} = \frac{n-r+1}{r} x$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then $T_{r+1} \geq T_r$ or $\frac{T_{r+1}}{T_r} \geq 1$.

$$\therefore \frac{n-r+1}{r} |x| \geq 1 \quad \text{or} \quad r \leq \frac{(n+1)}{(1+|x|)} |x| \quad \dots(i)$$

Now substituting values of n and x in (i), we get $r \leq m + f$ or $r \leq m$, where m is a positive integer and f is a fraction such that $0 < f < 1$.

When n is even T_{m+1} is the greatest term, when n is odd T_m and T_{m+1} are the greatest terms and both are equal.

To find the greatest term (numerically) in the expansion of $(1 + x)^n$.

$$(i) \quad \text{Calculate } m = \left\lfloor \frac{x(n+1)}{x+1} \right\rfloor$$

- (ii) If m is integer, then T_m and T_{m+1} are equal and both are greatest term.

- (iii) If m is not integer, then $T_{[m]+1}$ is the greatest term, where $[\cdot]$ denotes the greatest integral part.

Greatest coefficient

- (i) If n is even, then greatest coefficient is ${}^nC_{n/2}$.
- (ii) If n is odd, then greatest coefficient are ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n+3}{2}}$

PROPERTIES OF BINOMIAL COEFFICIENTS

In the binomial expansion of $(1 + x)^n$,

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n \quad \dots(i)$$

Let us denote the coefficients, ${}^nC_0, {}^nC_1, {}^nC_2, \dots$

by $C_0, C_1, C_2, \dots, C_r, \dots, C_n$ respectively then the above expression can be expressed as

$$(1 + x)^n = \sum_{r=0}^n C_r x^r = C_0 + C_1 x + C_2 x^2 + \dots + C_r x^r + \dots + C_n x^n.$$

where $C_0, C_1, C_2, \dots, C_n$ are known as binomial coefficients.

PROPERTIES

- $2^n = C_0 + C_1 + C_2 + \dots + C_n$
- $C_0 - C_1 + C_2 - C_3 + \dots = 0$
- $C_1 + C_3 + C_5 + \dots = C_2 + C_4 + C_6 + \dots = 2^{n-1}$
- ${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} {}^{n-2}C_{r-2}$ and so on.
- $C_0 C_r + C_1 C_{r+1} + \dots + C_{n-r} C_n = \frac{2n!}{(n-r)! (n+r)!}$
- $2^n C_n = C_0^2 + C_1^2 + \dots + C_n^2 = \frac{2n!}{(n!)^2}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $C_1 + 2C_2 + 3C_3 + \dots + n \cdot C_n = n \cdot 2^{n-1}$
- $C_1 - 2C_2 + 3C_3 - \dots = 0$
- $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = (n+2)2^{n-1}$
- $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$

USE OF DIFFERENTIATION AND INTEGRATION IN BINOMIAL THEOREM

- **Use of differentiation** : This method applied only when the numerals occur as the product of binomial coefficients.
- (i) If last term of the series leaving the plus or minus sign be m , then divide m by n if q be the quotient and r be the remainder. i.e., $m = nq + r$

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Then replace x by x^q in the given series and multiplying both sides of expansion by x^r .

(ii) After process (i), differentiate both sides, w.r.t. x and put $x = 1$ or -1 or i or $-i$ etc. according to given series.

(iii) If product of two numerals (or square of numerals) or three numerals (or cube of numerals) then differentiate twice or thrice.

- **Use of integration :** This method is applied only when the numerals occur as the denominator of the binomial coefficients.

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then we integrate both sides between the suitable limits which gives the required series.

- If the sum contains $C_0, C_1, C_2, \dots, C_n$ with all +ve signs, then integrate between limit 0 to 1.
- If the sum contains alternate signs then integrate between limit -1 to 0.
- If the sum contains odd coefficients i.e., (C_0, C_2, C_4, \dots) then integrate between -1 to 1.
- If the sum contains even coefficients (i.e., C_1, C_3, C_5, \dots) then subtracting (ii) from (i) and then dividing by 2.
- If denominator of binomial coefficients is product of two numerals then integrate two times, first taking limit between 0 to x and second time take suitable limits.

AN APPLICATION OF BINOMIAL THEOREM

If $(\sqrt{A} + B)^n = I + f$ where I and n are positive integers, $0 \leq f < 1$ then $(I + f) \cdot f = K^n$

where $A - B^2 = K > 0$ and $\sqrt{A} - B < 1$.

- If n is even integer then
 $(\sqrt{A} + B)^n + (\sqrt{A} - B)^n = I + f + f'$
Hence L.H.S. and I are integers
 $\therefore f + f'$ is also integer
 $\Rightarrow f + f' = 1$; $\therefore f' = (1 - f)$
Hence, $(I + f)(I - f) = (I + f)f'$
 $= (\sqrt{A} + B)^n (\sqrt{A} - B)^n = (A - B^2)^n = K^n$

MULTINOMIAL THEOREM (FOR POSITIVE INTEGRAL INDEX)

If n is positive integer and $a_1, a_2, a_3, \dots, a_n \in C$ then

$$(a_1 + a_2 + a_3 + \dots + a_m)^n = \sum \frac{n!}{n_1! n_2! n_3! \dots n_m!} a_1^{n_1} \cdot a_2^{n_2} \dots a_m^{n_m}$$

where $n_1, n_2, n_3, \dots, n_m$ are all non-negative integers subject to the condition, $n_1 + n_2 + n_3 + \dots + n_m = n$

- The coefficient of $a_1^{n_1} \cdot a_2^{n_2} \dots a_m^{n_m}$ in the expansion of $(a_1 + a_2 + a_3 + \dots + a_m)^n$ is $\frac{n!}{n_1! n_2! n_3! \dots n_m!}$

BINOMIAL THEOREM FOR ANY INDEX

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots \text{terms up to } \infty$$

when n is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible. If first term is not 1, then make first term unity in the

following way, $(x+y)^n = x^n \left[1 + \frac{y}{x} \right]^n$, if $\left| \frac{y}{x} \right| < 1$.

$$\text{General term : } T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

SOME IMPORTANT EXPANSIONS

- $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}x^r + \dots$
- $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}(-x)^r + \dots$

THREE/FOUR CONSECUTIVE TERMS OR COEFFICIENTS

- **If consecutive coefficients are given:** In this case divide consecutive coefficients pair wise. We get equations and then solve them.
- **If consecutive terms are given :** In this case divide consecutive terms pair wise i.e. if four consecutive terms be $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ then find $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$ (say) then divide λ_1 by λ_2 and λ_2 by λ_3 and solve.

PASCAL'S TRIANGLE

							$(x+y)^0$
						1	$(x+y)^1$
			1	2	1		$(x+y)^2$
		1	3	3	1		$(x+y)^3$
	1	4	6	4	1		$(x+y)^4$
1	5	10	10	5	1		$(x+y)^5$

Pascal's triangle gives the direct binomial coefficients.

Example : $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Let $p(n)$ be a statement involving the natural number n such that

- $p(1)$ is true i.e. $p(n)$ is true for $n = 1$.
- $p(m + 1)$ is true, whenever $p(m)$ is true i.e. $p(m + 1)$ is true.

Then $p(n)$ is true for all natural numbers n .

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

Let $p(n)$ be a statement involving the natural number n such that

- $p(1)$ is true i.e. $p(n)$ is true for $n = 1$ and
- $p(m + 1)$ is true, whenever $p(n)$ is true for all n , where $1 \leq n \leq m$.

Then $p(n)$ is true for all natural numbers.

DIVISIBILITY

To show that an expression is divisible by an integer

- If a, p, n, r are positive integers, then first of all we write $a^{pn+r} = a^{pn} \cdot a^r = (a^p)^n \cdot a^r$.
- If we have to show that the given expression is divisible by c .

Then express, $a^p = [1 + (a^p - 1)]$, if some power of $(a^p - 1)$ has c as a factor. $a^p = [2 + (a^p - 2)]$, if some power of $(a^p - 2)$ has c as a factor.

$a^p = [k + (a^p - k)]$, if some power of $(a^p - k)$ has c as a factor.

PROBLEMS

Single Correct Answer Type

- In the expansion of the following expression $1 + (1 + x) + (1 + x)^2 + \dots + (1 + x)^n$, the coefficient of x^k ($0 \leq k \leq n$) is
 - ${}^{n+1}C_{k+1}$
 - nC_k
 - ${}^nC_{n-k-1}$
 - none of these
- If $T_0, T_1, T_2, \dots, T_n$ represent the terms in the expansion of $(x + a)^n$, then

$$(T_0 - T_2 + T_4 - \dots)^2 + (T_1 - T_3 + T_5 - \dots)^2$$
 - $(x^2 + a^2)$
 - $(x^2 + a^2)^n$
 - $(x^2 + a^2)^{1/n}$
 - $(x^2 + a^2)^{-1/n}$
- The greatest integer which divides the number $101^{100} - 1$, is
 - 100
 - 1000
 - 10000
 - 100000
- The larger of $99^{50} + 100^{50}$ and 101^{50} is
 - $99^{50} + 100^{50}$
 - both are equal
 - 101^{50}
 - none of these
- The last digit in the expansion of 7^{300} is
 - 7
 - 9
 - 1
 - 3
- If the ratio of the coefficient of third and fourth term in the expansion of $\left(x - \frac{1}{2x}\right)^n$ is 1 : 2, then the value of n will be
 - 18
 - 16
 - 12
 - 10
- If the $(r + 1)^{\text{th}}$ term in the expansion of $\left(\sqrt[3]{\frac{a}{b}} + \sqrt{\frac{b}{3a}}\right)^{21}$ has the same power of a and b , then the value of r is
 - 9
 - 10
 - 8
 - 6
- If the third term in the binomial expansion of $(1 + x)^m$ is $-\frac{1}{8}x^2$, then the rational value of m is
 - 2
 - $\frac{1}{2}$
 - 3
 - 4
- The first 3 terms in the expansion of $(1 + ax)^n$, ($n \neq 0$) are 1, $6x$ and $16x^2$. Then the value of a and n are respectively
 - 2 and 9
 - 3 and 2
 - $2/3$ and 9
 - $3/2$ and 6
- If the coefficients of T_r, T_{r+1}, T_{r+2} terms of $(1 + x)^{14}$ are in A.P., then $r =$
 - 6
 - 7
 - 8
 - 9
- If the coefficients of $p^{\text{th}}, (p + 1)^{\text{th}}$ and $(p + 2)^{\text{th}}$ terms in the expansion of $(1 + x)^n$ are in A.P., then
 - $n^2 - 2np + 4p^2 = 0$
 - $n^2 - n(4p + 1) + 4p^2 - 2 = 0$
 - $n^2 - n(4p + 1) + 4p^2 = 0$
 - none of these
- If A and B are the coefficients of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then
 - $A = B$
 - $A = 2B$
 - $2A = B$
 - none of these
- In the expansion of $\left(x - \frac{1}{x}\right)^6$, the constant term is
 - 20
 - 20
 - 30
 - 30
- If the coefficients of $5^{\text{th}}, 6^{\text{th}}$ and 7^{th} terms in the expansion of $(1 + x)^n$ be in A.P., then $n =$

- (a) 7 only (b) 14 only
(c) 7 or 14 (d) none of these
15. The coefficient of x^{53} in the following expansion $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} \cdot 2^m$ is
(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$
(c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$
16. The coefficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is
(a) ${}^{15}C_5$ (b) ${}^{15}C_6$ (c) ${}^{15}C_4$ (d) ${}^{15}C_7$
17. If in the expansion of $(1+x)^m (1-x)^n$, the coefficient of x and x^2 are 3 and -6 respectively, then m is
(a) 6 (b) 9 (c) 12 (d) 24
18. If x^m occurs in the expansion of $\left(x + \frac{1}{x^2}\right)^{2n}$, then the coefficient of x^m is
(a) $\frac{(2n)!}{(m)!(2n-m)!}$ (b) $\frac{(2n)!3!3!}{(2n-m)!}$
(c) $\frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$
(d) none of these
19. If coefficients of 2^{nd} , 3^{rd} and 4^{th} terms in the binomial expansion of $(1+x)^n$ are in A.P., then $n^2 - 9n$ is equal to
(a) -7 (b) 7 (c) 14 (d) -14
20. In the expansion of $(1+x+x^3+x^4)^{10}$, the coefficient of x^4 is
(a) ${}^{40}C_4$ (b) ${}^{10}C_4$ (c) 210 (d) 310
21. The coefficient of x^5 in the expansion of $(1+x)^{21} + (1+x)^{22} + \dots + (1+x)^{30}$ is
(a) ${}^{51}C_5$ (b) 9C_5
(c) ${}^{31}C_6 - {}^{21}C_6$ (d) ${}^{30}C_5 + {}^{20}C_5$
22. The coefficient of x^{-9} in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^9$ is
(a) 512 (b) -512 (c) 521 (d) 251
23. If the second, third and fourth term in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, then the value of n is
(a) 15 (b) 20 (c) 10 (d) 5
24. In the expansion of $(1+x)^n$ the coefficient of p^{th} and $(p+1)^{\text{th}}$ terms are respectively p and q . Then $p+q =$
(a) $n+3$ (b) $n+1$
(c) $n+2$ (d) n
25. If in the expansion of $(1+x)^{21}$, the coefficients of x^r and x^{r+1} be equal, then r is equal to
(a) 9 (b) 10 (c) 11 (d) 12
26. In the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$, the term independent of x is
(a) ${}^9C_3 \cdot \frac{1}{6^3}$ (b) ${}^9C_3 \left(\frac{3}{2}\right)^3$
(c) 9C_3 (d) none of these
27. The greatest term in the expansion of $\sqrt{3}\left(1 + \frac{1}{\sqrt{3}}\right)^{20}$ is
(a) $\frac{25840}{9}$ (b) $\frac{24840}{9}$
(c) $\frac{26840}{9}$ (d) none of these
28. If n is even positive integer, then the condition that the greatest term in the expansion of $(1+x)^n$ may have the greatest coefficient also, is
(a) $\frac{n}{n+2} < x < \frac{n+2}{n}$ (b) $\frac{n+1}{n} < x < \frac{n}{n+1}$
(c) $\frac{n}{n+4} < x < \frac{n+4}{4}$ (d) none of these
29. The coefficient of $1/x$ in the expansion of $(1+x)^n \left(1 + \frac{1}{x}\right)^n$ is
(a) $\frac{n!}{(n-1)!(n+1)!}$ (b) $\frac{(2n)!}{(n-1)!(n+1)!}$
(c) $\frac{2n!}{(2n-1)!(2n+1)!}$ (d) none of these
30. The coefficient of x^5 in the expansion of $(x^2 - x - 2)^5$ is
(a) -83 (b) -82 (c) -81 (d) 0
31. The greatest coefficient in the expansion of $(1+x)^{2n+1}$ is
(a) $\frac{(2n+1)!}{n!(n+1)!}$ (b) $\frac{(2n+2)!}{n!(n+1)!}$
(c) $\frac{(2n+1)!}{[(n+1)!]^2}$ (d) $\frac{(2n)!}{(n!)^2}$

32. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \dots + C_{n-r}C_n =$
 (a) $\frac{(2n)!}{(n-r)!(n+r)!}$ (b) $\frac{n!}{(-r)!(n+r)!}$
 (c) $\frac{n!}{(n-r)!}$ (d) none of these
33. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then
 $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \dots + C_n^2 =$
 (a) $\frac{n!}{n!n!}$ (b) $\frac{(2n)!}{n!n!}$
 (c) $\frac{(2n)!}{n!}$ (d) None of these
34. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then
 $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \dots + \frac{nC_n}{C_{n-1}} =$
 (a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+2)}{2}$
 (c) $\frac{n(n+1)}{2}$ (d) $\frac{(n-1)(n-2)}{2}$
35. If a and d are two complex numbers, then the sum to $(n+1)$ terms of the following series $aC_0 - (a+d)C_1 + (a+2d)C_2 - \dots$ is
 (a) $\frac{a}{2^n}$ (b) na
 (c) 0 (d) none of these
36. If $(1+x)^{15} = C_0 + C_1x + C_2x^2 + \dots + C_{15}x^{15}$, then
 $C_2 + 2C_3 + 3C_4 + \dots + 14C_{15} =$
 (a) $14 \cdot 2^{14}$ (b) $13 \cdot 2^{14} + 1$
 (c) $13 \cdot 2^{14} - 1$ (d) none of these
37. If $x + y = 1$, then $\sum_{r=0}^n r^2 {}^nC_r x^r y^{n-r}$ equals
 (a) nxy (b) $nx(x+yn)$
 (c) $nx(nx+y)$ (d) none of these
38. For $r = 0, 1, \dots, 10$, let A_r , B_r and C_r denote, respectively, the coefficient of x^r in the expansion of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$, then
 $\sum_{r=1}^{10} A_r (B_{10}B_r - C_{10}A_r)$ is equal to
 (a) $C_{10} - B_{10}$
 (b) 0
 (c) $A_{10} (B_{10}^2 - C_{10}A_{10})$
 (d) $B_{10} - C_{10}$

Multiple Correct Answer Type

39. The value of $\sum_{k=0}^7 \left[\frac{\binom{7}{k}}{\binom{14}{k}} \sum_{r=k}^{14} \binom{r}{k} \binom{14}{r} \right]$, where $\binom{n}{r}$ denotes nC_r , is
 (a) 6^7 (b) greater than 7^6
 (c) 8^7 (d) greater than 7^8
40. The largest coefficient in the expansion of $(4+3x)^{25}$ is
 (a) ${}^{25}C_{11} 3^{25} \left(\frac{4}{3}\right)^{14}$ (b) ${}^{25}C_{11} 4^{25} \left(\frac{3}{4}\right)^{11}$
 (c) $C_{14} 4^{14} 3^{11}$ (d) ${}^{25}C_{14} 4^{11} 3^{14}$
41. Let $R = (8+3\sqrt{7})^{20}$ and $[R]$ = the greater integer less than or equal to R . Then
 (a) $[R]$ is even
 (b) $[R]$ is odd
 (c) $R - [R] = 1 - \frac{1}{(8+3\sqrt{7})^{20}}$
 (d) $R + R[R] = 1 + R^2$
42. If the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000 then x is equal to
 (a) 100 (b) 10 (c) 1 (d) $\frac{1}{\sqrt{10}}$
43. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then
 (a) $a_0 - a_2 + a_4 - a_6 + \dots = 0$, if n is odd
 (b) $a_1 - a_3 + a_5 - a_7 + \dots = 0$, if n is even
 (c) $a_0 - a_2 + a_4 - a_6 + \dots = 0$, if $n = 4p$, $p \in \mathbb{I}^+$
 (d) $a_1 - a_3 + a_5 - a_7 + \dots = 0$, if $n = 4p + 1$, $p \in \mathbb{I}^+$
44. If n is a positive integer and $(3\sqrt{3}+5)^{2n+1} = \alpha + \beta$ where α is an integer and $0 < \beta < 1$ then
 (a) α is an even integer
 (b) $(\alpha + \beta)^2$ is divisible by 2^{2n+1}
 (c) The integer just below $(3\sqrt{3}+5)^{2n+1}$ divisible by 3
 (d) α is divisible by 10
45. Let $a_n = \left(1 + \frac{1}{n}\right)^n$ then for each $n \in \mathbb{N}$
 (a) $a_n \geq 2$ (b) $a_n < 3$ (c) $a_n < 4$ (d) $a_n < 2$

Comprehension Type

Paragraph for Q. No. 46 to 48

$$\text{If } (1+x+x^2)^{100} = \sum_{r=0}^{200} a_r x^r$$

46. Which of the following is true?

- (a) $a_{28} = a_{72}$ (b) $a_{56} = a_{144}$
(c) $a_{200} = a_{300}$ (d) none of these

47. $a_0 + a_1 + a_2 + \dots + a_{99}$ is equal to

- (a) $\frac{3^{99} - a_{99}}{2}$ (b) $\frac{3^{101} - a_{99}}{2}$
(c) $\frac{3^{100} - a_{100}}{2}$ (d) none of these

48. $37a_{37}$ is equal to

- (a) $64a_{36} + 105a_{35}$ (b) $64a_{35} + 148a_{36}$
(c) $56a_{32} + 168a_{22}$ (d) none of these

Paragraph for Q. No. 49 to 50

Given that $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ where x is complex number and C_0, C_1, \dots, C_n are constants. Then,

49. The value of $C_0 + C_3 + C_6 + \dots$ will be

- (a) $\frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right)$ (b) $\frac{1}{3} \left(2^n - 2 \cos \frac{n\pi}{3} \right)$
(c) $\frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{4} \right)$ (d) $\frac{1}{3} \left(2^n - 2 \cos \frac{n\pi}{4} \right)$

50. The value of $C_2 + C_5 + C_8 + \dots$ will be

- (a) $\frac{1}{3} \left[2^n + 2^2 \cos(n-2) \frac{\pi}{3} \right]$
(b) $\frac{1}{3} \left[2^n - 2 \cos(n-2) \frac{\pi}{3} \right]$
(c) $\frac{1}{3} \left[2^n - 2^2 \cos(n-2) \frac{\pi}{3} \right]$
(d) $\frac{1}{3} \left[2^n + 2 \cos(n+2) \frac{\pi}{3} \right]$

Paragraph for Q. No. 51 to 52

Integration plays a vital role in proving identities involving binomial coefficients whose algebraic method of proving is in general cumbersome and requires the help of mathematical induction. If we apply integration techniques, several binomial identities are easily proved. For instance

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + \frac{(-1)^n C_n}{n+1} = \int_0^1 (1-x)^n dx = \frac{1}{n+1},$$

where $C_r = {}^nC_r$, $r = 0, 1, 2, \dots, n$

51. If $S_1 = \sum_{k=0}^n \frac{(-1)^k {}^nC_k}{k+m+1}$, $S_2 = \sum_{k=0}^m \frac{(-1)^k {}^mC_k}{k+n+1}$, where $m > n$ then

- (a) $S_2 = \frac{m+n}{m-n} S_1$ (b) $S_2 = -S_1$
(c) $S_2 = S_1$
(d) $S_1 = S_2 = 0$ for all m and n

52. The value of binomial series

$$C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + \frac{(-1)^{n-1} C_n}{n}$$
 must be equal to

- (a) $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$
(b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n-1}}{n}$
(c) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$
(d) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + \frac{(-1)^{n-1}}{n^2}$

Matrix-Match Type

53. Match the following.

Column-I		Column-II	
(A)	$\sum_{0 \leq i < j \leq n} (i+j)(C_i \cdot C_j) =$	(p)	$n \cdot 2^n$
(B)	$\sum_{0 \leq i < j \leq n} \sum ({}^nC_i + {}^nC_j) =$	(q)	$(n+1)2^n C_n$
(C)	$\sum_{0 \leq i < j \leq n} \sum i \cdot {}^nC_j =$	(r)	$n(n-1)2^{n-3}$
		(s)	$\frac{n}{2} \cdot [2^{2n} - 2^n C_n]$

54. Match the following.

Column-I		Column-II	
(A)	Number of distinct terms in the expansion of $(x+y-z)^{16}$ is	(p)	2^{12}
(B)	Number of terms in the expansion of $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$ is	(q)	97

(C)	The number of irrational terms in $(\sqrt[8]{5} + \sqrt[6]{2})^{100}$ is	(r)	4
(D)	The sum of numerical coefficients in the expansion of $\left(1 + \frac{x}{3} + \frac{2y}{3}\right)^{12}$ is	(s)	153

55. Consider the binomial expansion of $(1 + 2x)^n$ (n is a positive integer) in which the sum of the coefficients is 6561. Let $(1 + 2x)^n = R = I + f$ where I is the largest integer not exceeding R and $0 < f < 1$.

Column-I		Column-II	
(A)	If r^{th} term in the expansion is the greatest term, then r cannot exceed	(p)	3
(B)	If i^{th} term is having the greatest coefficient, then i can be	(q)	4
(C)	The number of integral terms in the expansion when $x = \sqrt[3]{3}$ is less than	(r)	5
(D)	For $x = \frac{1}{\sqrt{2}}$, the value of $R(1 - f)$ is less than	(s)	6
		(t)	7

Integer Answer Type

56. The sum of last 3 digits of 3^{100} is

57. If $n \in N$, and $C_k = {}^nC_k$ and

$$\sum_{k=1}^n k^3 \left(\frac{{}^nC_k}{{}^nC_{k-1}} \right)^2 = \frac{n(n+1)^2(n+2)}{3p} \text{ then } p \text{ is}$$

58. Given ${}^8C_1 x(1-x)^7 + 2 \cdot {}^8C_2 x^2(1-x)^6 + 3 \cdot {}^8C_3 x^3(1-x)^5 + \dots + 8 \cdot x^8 = ax + b$

Find $a + b$.

59. If $\sum_{r=0}^n (-1)^r \frac{{}^nC_r}{(r+2)C_r} = \frac{k}{n+2}$, find k .

60. Find the coefficient of x^{103} in $(1 + x + x^2 + x^3 + x^4)^{199} (x - 1)^{201}$.

SOLUTIONS

1. (a) : Given expression

$$E = 1 + (1+x) + (1+x)^2 + \dots + (1+x)^n, \text{ is in G. P.}$$

$$\therefore E = \frac{(1+x)^{n+1} - 1}{(1+x) - 1} = x^{-1} \{(1+x)^{n+1} - 1\}$$

$$\therefore \text{Coefficient of } x^k \text{ in } E = \text{Coefficient of } x^{k+1} \text{ in } \{(1+x)^{n+1} - 1\} = {}^{n+1}C_{k+1}$$

2. (b) : From the given condition, replacing a by ai and $-ai$ respectively, we get

$$(x + ai)^n = (T_0 - T_2 + T_4 - \dots) + i(T_1 - T_3 + T_5 - \dots) \quad \dots(i)$$

$$\text{and } (x - ai)^n = (T_0 - T_2 + T_4 - \dots) - i(T_1 - T_3 + T_5 - \dots)$$

$$\dots(ii)$$

Multiplying (ii) and (i) we get required result

$$\text{i.e., } (x^2 + a^2)^n = (T_0 - T_2 + T_4 - \dots)^2 + (T_1 - T_3 + T_5 - \dots)^2$$

$$\begin{aligned} 3. (c) : (1+100)^{100} &= 1 + 100 \cdot 100 + \frac{100 \cdot 99}{1 \cdot 2} \cdot (100)^2 \\ &\quad + \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3} (100)^3 + \dots \\ (101)^{100} - 1 &= 100 \cdot 100 \left[1 + \frac{100 \cdot 99}{1 \cdot 2} + \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3} \cdot 100 + \dots \right] \end{aligned}$$

From above it is clear that,

$$(101)^{100} - 1 \text{ is divisible by } (100)^2 = 10000$$

$$\begin{aligned} 4. (c) : \text{We have, } 101^{50} &= (100+1)^{50} \\ &= 100^{50} + 50 \cdot 100^{49} + \frac{50 \cdot 49}{2 \cdot 1} 100^{48} + \dots \quad \dots(i) \end{aligned}$$

$$\text{and } 99^{50} = (100-1)^{50}$$

$$= 100^{50} - 50 \cdot 100^{49} + \frac{50 \cdot 49}{2 \cdot 1} 100^{48} - \dots \quad \dots(ii)$$

Subtracting (ii) from (i), we get

$$101^{50} - 99^{50} = 100^{50} + 2 \frac{50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3} 100^{47} > 100^{50}$$

Hence $101^{50} > 100^{50} + 99^{50}$.

$$\begin{aligned} 5. (c) : \text{We have, } 7^{300} &= (7^2)^{150} = (50 - 1)^{150} \\ &= {}^{150}C_0 (50)^{150} (-1)^0 + {}^{150}C_1 (50)^{149} (-1)^1 + \dots \end{aligned}$$

$$+ {}^{150}C_{150} (50)^0 (-1)^{150}$$

Thus the last digits of 7^{300} are ${}^{150}C_{150} \cdot 1 \cdot 1$ i.e., 1.

$$6. (d) : T_3 = {}^nC_2 (x)^{n-2} \left(-\frac{1}{2x} \right)^2; T_4 = {}^nC_3 (x)^{n-3} \left(-\frac{1}{2x} \right)^3$$

But according to the condition,

$$\frac{-n(n-1) \times 3 \times 2 \times 1 \times 8}{n(n-1)(n-2) \times 2 \times 1 \times 4} = \frac{1}{2} \Rightarrow n = -10$$

7. (a) : We have $T_{r+1} = {}^{21}C_r \left(3\sqrt{\frac{a}{\sqrt{b}}} \right)^{21-r} \left(\sqrt{\frac{b}{\sqrt{a}}} \right)^r$
 $= {}^{21}C_r a^{7-(r/2)} b^{(2/3)r-(7/2)}$

Since the powers of a and b are the same,

$$\therefore 7 - \frac{r}{2} = \frac{2}{3}r - \frac{7}{2} \Rightarrow r = 9$$

8. (b) : We have, $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots$

By hypothesis, $\frac{m(m-1)}{2}x^2 = -\frac{1}{8}x^2$

$$\Rightarrow 4m^2 - 4m = -1 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}$$

9. (c) : We have, $T_1 = {}^nC_0 = 1$... (i)

$T_2 = {}^nC_1 ax = 6x$... (ii)

$T_3 = {}^nC_2 (ax)^2 = 16x^2$... (iii)

From (ii), $\frac{n!}{(n-1)!}a = 6 \Rightarrow na = 6$... (iv)

From (iii), $\frac{n(n-1)}{2}a^2 = 16$... (v)

Only (c) is satisfying equation (iv) and (v).

10. (d)

11. (b) : Coefficient of $p^{\text{th}}, (p+1)^{\text{th}}$ and $(p+2)^{\text{th}}$ terms in expansion of $(1+x)^n$ are ${}^nC_{p-1}, {}^nC_p, {}^nC_{p+1}$.

Then $2^n C_p = {}^nC_{p-1} + {}^nC_{p+1}$

$$\Rightarrow n^2 - n(4p+1) + 4p^2 - 2 = 0$$

12. (b) : $\frac{\text{Coefficient of } x^n \text{ in expansion of } (1+x)^{2n}}{\text{Coefficient of } x^n \text{ in expansion of } (1+x)^{2n-1}}$

$$= \frac{{}^{2n}C_n}{{}^{(2n-1)}C_n} = \frac{(2n)!}{n!n!} \times \frac{(n-1)!n!}{(2n-1)!} = \frac{2n}{n} = 2:1$$

$$\Rightarrow \frac{A}{B} = \frac{2}{1} \Rightarrow A = 2B$$

13. (a) : In the expansion of $\left(x - \frac{1}{x}\right)^6$, the general term

is ${}^6C_r x^{6-r} \left(-\frac{1}{x}\right)^r = {}^6C_r (-1)^r x^{6-2r}$

For term independent of x , $6-2r=0 \Rightarrow r=3$
 Thus the required coefficient $= (-1)^3 \cdot {}^6C_3 = -20$.

14. (c) : Coefficient of $T_5 = {}^nC_4, T_6 = {}^nC_5$ and $T_7 = {}^nC_6$
 According to the condition, $2 {}^nC_5 = {}^nC_4 + {}^nC_6$

$$\Rightarrow 2 \left[\frac{n!}{(n-5)!5!} \right] = \left[\frac{n!}{(n-4)!4!} + \frac{n!}{(n-6)!6!} \right]$$

$$\Rightarrow 2 \left[\frac{1}{(n-5)5} \right] = \left[\frac{1}{(n-4)(n-5)} + \frac{1}{6 \times 5} \right]$$

After solving, we get $n = 7$ or 14 .

15. (c) : The given expansion is $[(x-3) + 2]^{100}$
 $= (x-1)^{100} = (1-x)^{100}$

$\therefore x^{53}$ will occur in T_{54} i.e., $T_{54} = {}^{100}C_{53}(-x)^{53}$

\therefore Coefficient is $-{}^{100}C_{53}$.

16. (c) : Let T_{r+1} term containing x^{32} .

Therefore ${}^{15}C_r x^{4r} \left(\frac{-1}{x^3}\right)^{15-r}$

$$\Rightarrow x^{4r} x^{-45+3r} = x^{32} \Rightarrow 7r = 77 \Rightarrow r = 11$$

Hence coefficient of x^{32} is ${}^{15}C_{11}$ or ${}^{15}C_4$

17. (c)

18. (c) : $T_{r+1} = {}^{2n}C_r x^{2n-r} \left(\frac{1}{x^2}\right)^r = {}^{2n}C_r x^{2n-3r}$,

This contains x^m , if $2n-3r = m$ i.e., if $r = \frac{2n-m}{3}$

$$\therefore \text{Coefficient of } x^m = {}^{2n}C_r, r = \frac{2n-m}{3}$$

$$= \frac{2n!}{(2n-r)!r!} = \frac{2n!}{\left(2n - \frac{2n-m}{3}\right)! \left(\frac{2n-m}{3}\right)!}$$

$$= \frac{2n!}{\left(\frac{4n+m}{3}\right)! \left(\frac{2n-m}{3}\right)!}$$

19. (d) : Coefficients of 2nd, 3rd and 4th terms are respectively ${}^nC_1, {}^nC_2$ and nC_3 are in A.P.

$$\Rightarrow 2 \cdot {}^nC_2 = {}^nC_1 + {}^nC_3$$

$$\Rightarrow \frac{2n!}{2!(n-2)!} = \frac{n!}{(n-1)!} + \frac{n!}{3!(n-3)!}$$

On solving, $n^2 - 9n + 14 = 0 \Rightarrow n^2 - 9n = -14$.

20. (d) : $(1+x+x^3+x^4)^{10} = (1+x)^{10} (1+x^3)^{10}$
 $= (1+{}^{10}C_1 \cdot x + {}^{10}C_2 \cdot x^2 + \dots)(1+{}^{10}C_1 \cdot x^3 + {}^{10}C_2 \cdot x^6 + \dots)$

\therefore Coefficient of $x^4 = {}^{10}C_1 \cdot {}^{10}C_1 + {}^{10}C_4 = 310$

21. (c)

22. (b): Here $T_{r+1} = {}^9C_r \left(\frac{x^2}{2} \right)^{9-r} \left(\frac{-2}{x} \right)^r$

$$= {}^9C_r \frac{x^{18-3r}(-2)^r}{2^{9-r}},$$

this contains x^{-9} if $18 - 3r = -9$ i.e. if $r = 9$.

\therefore Coefficient of $x^{-9} = {}^9C_9 \frac{(-2)^9}{2^0} = -2^9 = -512$

23. (d): $T_2 = n(x)^{n-1}(a)^1 = 240$... (i)

$T_3 = \frac{n(n-1)}{1 \cdot 2} x^{n-2} a^2 = 720$... (ii)

$T_4 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} a^3 = 1080$... (iii)

To eliminate x ,

$$\frac{T_2 \cdot T_4}{T_3^2} = \frac{240 \cdot 1080}{720 \cdot 720} = \frac{1}{2} \Rightarrow \frac{T_2}{T_3} \cdot \frac{T_4}{T_3} = \frac{1}{2}$$

Now, $\frac{T_{r+1}}{T_r} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

Putting $r = 3$ and 2 in above expression, we get

$$\Rightarrow \frac{n-2}{3} \cdot \frac{2}{n-1} = \frac{1}{2} \Rightarrow n = 5.$$

24. (b): p^{th} term $= T_p = {}^nC_{p-1}(x)^{n-p+1}(1)^{p-1} = p$

$(p+1)^{\text{th}}$ term $= T_{p+1} = {}^nC_p(x)^{n-p}(1)^p = q$

Now, $\frac{p}{q} = \frac{{}^nC_{p-1}}{{}^nC_p}$

25. (b): $T_{r+1} = {}^{21}C_r(1)^{21-r}(x)^r = {}^{21}C_r$

\therefore Coefficient of $x^r = {}^{21}C_r$ and
 coefficient of $x^{r+1} = {}^{21}C_{r+1}$

So, we must have ${}^{21}C_r = {}^{21}C_{r+1} \Rightarrow r = 10$.

26. (a): General term is $T_{r+1} = {}^9C_r \cdot \left(\frac{3x^2}{2} \right)^{9-r} \left(-\frac{1}{3x} \right)^r$

$$= {}^9C_r \left(\frac{3}{2} \right)^{9-r} \left(-\frac{1}{3} \right)^r x^{18-3r}$$

For the term independent of x , $18 - 3r = 0 \Rightarrow r = 6$

$$\therefore T_{6+1} = {}^9C_6 \left(\frac{3}{2} \right)^{9-6} \left(-\frac{1}{3} \right)^6 = {}^9C_3 \cdot \frac{1}{6^3}$$

27. (a)

28. (a)

29. (b): $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$

$$\left(1 + \frac{1}{x} \right)^n = {}^nC_0 + {}^nC_1 \frac{1}{x} + {}^nC_2 \frac{1}{x^2} + \dots + {}^nC_n \left(\frac{1}{x} \right)^n$$

Obviously, required coefficient of $\frac{1}{x}$ can be given by

$${}^nC_0 {}^nC_1 + {}^nC_1 {}^nC_2 + \dots + {}^nC_{n-1} {}^nC_n = \frac{(2n)!}{(n-1)!(n+1)!}$$

30. (c): $(x^2 - x - 2)^5 = (x-2)^5(1+x)^5$

$$= [{}^5C_0x^5 - {}^5C_1x^4 \times 2 + \dots][{}^5C_0 + {}^5C_1x + \dots]$$

\therefore Coefficient of x^5 :

$$1 - 5 \cdot 5 \cdot 2 + 10 \cdot 10 \cdot 4 - 10 \cdot 10 \cdot 8 + 5 \cdot 5 \cdot 16 - 32$$

$$= 1 - 50 + 400 - 800 + 400 - 32 = -81$$

31. (a): We know, $\frac{T_{r+1}}{T_r} = \frac{N-r+1}{r} \cdot x$

Given $N = 2n+1 \Rightarrow \frac{T_{r+1}}{T_r} = \frac{2n+2-r}{r} \cdot x$

$\therefore T_{r+1} \geq T_r$

$$\Rightarrow 2n+2-r \geq r \Rightarrow 2n+2 \geq 2r \Rightarrow r \leq n+1$$

$\therefore r = n$

$$T_{r+1} = T_{n+1} = {}^{2n+1}C_{n+1} = \frac{(2n+1)!}{(n+1)!n!}.$$

32. (a): $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_rx^r + \dots$... (i)

$$\left(1 + \frac{1}{x} \right)^n = C_0 + C_1 \frac{1}{x} + C_2 \frac{1}{x^2} + \dots + C_r \frac{1}{x^r} + \dots$$
 ... (ii)

Multiplying both sides and equating coefficient of x^r in $\frac{1}{x^n}(1+x)^{2n}$ or the coefficient of x^{n+r} in $(1+x)^{2n}$

we get the value of required expression

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$$

33. (b): $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$... (i)

and $\left(1 + \frac{1}{x} \right)^n = C_0 + C_1 \frac{1}{x} + C_2 \left(\frac{1}{x} \right)^2 + \dots + C_n \left(\frac{1}{x} \right)^n$... (ii)

If we multiply (i) and (ii), we get

$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$ is the term independent of x and hence it is equal to the term independent of x in

the product $(1+x)^n \left(1 + \frac{1}{x}\right)^n$ or in $\frac{1}{x^n}(1+x)^{2n}$ or term containing x^n in $(1+x)^{2n}$. Clearly the coefficient of x^n in $(1+x)^{2n}$ is T_{n+1} and equal to ${}^{2n}C_n = \frac{(2n)!}{n!n!}$

$$\begin{aligned} 34. (c) : & \frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \dots + n \cdot \frac{C_n}{C_{n-1}} \\ &= \frac{n}{1} + 2 \cdot \frac{n(n-1)/1 \cdot 2}{n} + 3 \cdot \frac{n(n-1)(n-2)/3 \cdot 2 \cdot 1}{n(n-1)/1 \cdot 2} + \dots + n \cdot \frac{1}{n} \\ &= n + (n-1) + (n-2) + \dots + 1 = \sum n = \frac{n(n+1)}{2} \end{aligned}$$

35. (c)

36. (b) : We have,

$$(1+x)^{15} = C_0 + C_1x + C_2x^2 + \dots + C_{15}x^{15}$$

$$\Rightarrow \frac{(1+x)^{15} - 1}{x} = C_1 + C_2x + \dots + C_{15}x^{14}$$

Differentiating both sides with respect to x , we get

$$= \frac{x \cdot 15(1+x)^{14} - (1+x)^{15} + 1}{x^2} = C_2 + 2C_3x + \dots + 14C_{15}x^{13}$$

Putting $x = 1$, we get

$$C_2 + 2C_3 + \dots + 14C_{15} = 15 \cdot 2^{14} - 2^{15} + 1 = 13 \cdot 2^{14} + 1.$$

37. (c)

$$\begin{aligned} 38. (a) : & \sum_{r=1}^{10} A_r (B_{10}B_r - C_{10}A_r) \\ &= \sum_{r=1}^{10} {}^{10}C_r ({}^{20}C_{10} {}^{20}C_r - {}^{30}C_{10} {}^{10}C_r) \\ &= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_{10-r} {}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_r {}^{10}C_{10-r} \\ &= {}^{20}C_{10} ({}^{30}C_{10} - 1) - {}^{30}C_{10} ({}^{20}C_{10} - 1) = {}^{30}C_{10} - {}^{20}C_{10} \\ &= C_{10} - B_{10} \end{aligned}$$

$$\begin{aligned} 39. (a,b) : & \sum_{k=0}^7 \left(\frac{{}^7C_k}{{}^{14}C_k} \sum_{r=k}^{14} {}^rC_k \cdot {}^{14}C_r \right) \\ &= \sum_{k=0}^7 \left(\frac{{}^7C_k}{{}^{14}C_k} \times \frac{|k|}{|14-k|} \sum_{r=k}^{14} \frac{|r|}{|k|} \cdot \frac{|14|}{|r-k|} \right) \\ &= \sum_{k=0}^7 \left({}^7C_k \sum_{r=k}^{14} {}^{14-k}C_{r-k} \right) = \sum_{k=0}^7 {}^7C_k \cdot 2^{14-k} \\ &= 2^{14} \sum_{k=0}^7 {}^7C_k \left(\frac{1}{2} \right)^k = 2^{14} \cdot \left(1 + \frac{1}{2} \right)^7 = 6^7 > 7^6 \end{aligned}$$

$$40. (a,b,c) : \text{We have, } (4+3x)^{25} = 4^{25} \left(1 + \frac{3}{4}x \right)^{25}$$

Let $(r+1)^{\text{th}}$ is the term having greatest coefficient.

\Rightarrow coefficient of $T_{r+1} \geq$ coefficient of T_r

$$\Rightarrow 4^{25} \left\{ {}^{25}C_r \left(\frac{3}{4} \right)^r \right\} \geq 4^{25} \left\{ {}^{25}C_{r-1} \left(\frac{3}{4} \right)^{r-1} \right\}$$

$$\Rightarrow \frac{{}^{25}C_r}{{}^{25}C_{r-1}} \left(\frac{3}{4} \right) \geq 1 \Rightarrow \frac{25-(r-1)}{r} \cdot \frac{3}{4} \geq 1$$

$$\Rightarrow 75 - 3r + 3 \geq 4r \therefore r \leq \frac{78}{7} \leq 11.142$$

But, r is an integer, hence $r = 11$

$$\begin{aligned} 41. (b,c,d) : R &= [R] + f = (8+3\sqrt{7})^{20} \\ &= {}^{20}C_0 8^{20} + {}^{20}C_1 8^{19} (3\sqrt{7}) + \dots, \end{aligned}$$

Where $0 < f < 1$

$$\text{Let } F = (8-3\sqrt{7})^{20} = {}^{20}C_0 8^{20} - {}^{20}C_1 8^{19} (3\sqrt{7}) + \dots$$

where $0 < F < 1$

$$\therefore [R] + f + F = 2[{}^{20}C_0 8^{20} + {}^{20}C_2 8^{18}$$

$$+ {}^{20}C_2 8^{16} (3\sqrt{7})^2 + \dots] = \text{an even integer}$$

$\therefore [R] = \text{an even integer} - 1 = \text{an odd integer}$

$$\text{Also, } R - [R] = f = 1 - F = 1 - (8-3\sqrt{7})^{20}$$

$$= 1 - \frac{1}{(8+3\sqrt{7})^{20}}$$

$$\text{Again } RF = 1 \Rightarrow R(1-f) = 1 \Rightarrow R\{1 - R + [R]\} = 1$$

$$\text{Therefore } R + R[R] = 1 + R^2$$

$$42. (a,d) : T_3 = {}^5C_2 \frac{1}{x^3} \cdot (x^{\log_{10} x})^2 = 10x^{-3+2\log_{10} x} = 1000$$

$$\therefore x^{-3+2\log_{10} x} = 100 \Rightarrow x = 100, \frac{1}{\sqrt{10}}$$

$$\begin{aligned} 43. (a,b) : & \therefore (1+x+x^2)^n \\ &= a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n} \end{aligned} \quad \dots(i)$$

Put $x = i$ in (i) we get,

$$(1+i+i^2)^n = (a_0 - a_2 + a_4 - a_6 + \dots) + i(a_1 - a_3 + a_5 - a_7 + \dots)$$

$$\Rightarrow i^n = (a_0 - a_2 + a_4 - a_6 + \dots) + i(a_1 - a_3 + a_5 - a_7 + \dots)$$

If n is odd, then $\text{Re}(i^n) = 0$

$$\Rightarrow a_0 - a_2 + a_4 - a_6 + \dots = 0$$

If n is even, then $\text{Im}(i^n) = 0$

$$\Rightarrow a_1 - a_3 + a_5 - a_7 + \dots = 0$$

44. (a,b,d)

$$45. (b, c) : a_n = 2 + \frac{1}{2} \left(1 - \frac{1}{n} \right) + \frac{1}{3} \left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) + \dots$$

$$+ \frac{1}{n} \left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) \dots \left(1 - \frac{n-1}{n} \right)$$

$$\text{Take } a_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$a_n \leq 3 - \frac{1}{2^{n-1}} < 3 \forall n \geq 1 \Rightarrow a_n < 4 \forall n \geq 1$$

46. (b) : Replacing x by $\frac{1}{x}$, we get

$$\sum_{r=0}^{200} a_r \left(\frac{1}{x} \right)^r = \left(1 + \frac{1}{x} + \frac{1}{x^2} \right)^{100} = \frac{1}{x^{200}} (x^2 + x + 1)$$

$$\Rightarrow \sum_{r=0}^{200} a_r x^{200-r} = (x^2 + x + 1)^{100} = \sum_{r=0}^{200} a_{200-r} x^{200-r}$$

Equating the coefficient of x^{200-r} , we get $a_r = a_{200-r}$

47. (c) : Put $x = 1$ in given expression, we get

$$a_0 + a_1 + a_2 + \dots + a_{200} = 3^{100}$$

$$\text{But } a_r = a_{200-r}$$

$$\therefore 2(a_0 + \dots + a_{99}) + a_{100} = 3^{100}$$

$$\Rightarrow a_0 + a_1 + \dots + a_{99} = \frac{1}{2}(3^{100} - a_{100})$$

48. (d) : On differentiating, we get

$$100(1+2x)(1+x+x^2)^{99} = \sum_{r=0}^{200} r a_r x^{r-1}$$

$$49. (a) : (1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

Put $x = 1, \omega, \omega^2$ and adding we get,

$$3(C_0 + C_3 + C_6 + \dots) = 2^n + (-\omega^2)^n + (-\omega)^n$$

$$-\omega = \frac{1}{2} - i \frac{\sqrt{3}}{2} = \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}; -\omega^2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$\therefore C_0 + C_3 + C_6 + \dots$$

$$= \frac{1}{3} \left[2^n + \cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} + \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right]$$

$$= \frac{1}{3} \left[2^n + 2 \cos \frac{n\pi}{3} \right]$$

$$50. (d) : (1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + C_7 x^7 + C_8 x^8 + \dots$$

Multiply by x on both sides we get,

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + C_4 x^5 + C_5 x^6 + C_6 x^7 + C_7 x^8 + C_8 x^9 + \dots$$

Put $x = 1, \omega, \omega^2$ and then adding we get

$$3 \cdot (C_2 + C_5 + C_8 + \dots) = 2^n + e^{\frac{2\pi i}{3}} \cdot e^{\frac{n\pi i}{3}} + e^{\frac{4\pi i}{3}} \cdot e^{\frac{n\pi i}{3}}$$

$$= 2^n + e^{\frac{(n+2)\pi i}{3}} + e^{-\frac{(n+2)\pi i}{3}} = 2^n + 2 \cos \frac{(n+2)\pi}{3}$$

$$\therefore C_2 + C_5 + C_8 + \dots = \frac{1}{3} \left[2^n + 2 \cos(n+2) \frac{\pi}{3} \right]$$

$$51. (c) : S_1 = \int_0^1 x^m (1-x)^n dx;$$

$$S_2 = \int_0^1 x^n (1-x)^m dx \Rightarrow S_1 = S_2$$

$$52. (a) : \text{Given series} = \int_0^1 \frac{1-(1-x)^n}{x} dx$$

$$= \int_0^1 \frac{1-x^n}{1-x} dx = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

53. (A) \rightarrow (s), (B) \rightarrow (p), (C) \rightarrow (r)

$$(A) S = \sum_{0 \leq i < j \leq n} (i+j)(C_i \cdot C_j) \\ = \sum_{0 \leq i < j \leq n} (n-i+n-j)(C_i \cdot C_j)$$

$$2S = 2n \sum_{0 \leq i < j \leq n} (C_i \cdot C_j)$$

$$S = n \sum C_i \cdot C_j = \frac{n}{2} [2^{2n} - 2n C_n]$$

$$\left[\because \sum_{0 \leq i < j \leq n} (C_i \cdot C_j) = \frac{1}{2} [2^{2n} - 2n C_n] \right]$$

$$(B) \sum_{0 \leq i < j \leq n} ({}^n C_i + {}^n C_j) = \sum_{0 \leq i < j \leq n} {}^n C_i + \sum_{0 \leq i < j \leq n} {}^n C_j \\ = n \cdot 2^{n-1} + n \cdot 2^{n-1} = n \cdot 2^n$$

$$(C) \sum_{0 \leq i < j \leq n} i \cdot {}^n C_j = \sum_{j=1}^n {}^n C_j (0+1+2+\dots+j-1)$$

$$= \sum_{j=1}^n {}^n C_j \frac{j(j-1)}{2} = \frac{1}{2} \sum_{j=1}^n j^2 {}^n C_j - \frac{1}{2} \sum_{j=1}^n j {}^n C_j$$

$$= \frac{1}{2} (n+1)n \cdot 2^{n-2} - \frac{1}{2} n \cdot 2^{n-1} = n(n-1) \cdot 2^{n-3}$$

54. (A) \rightarrow (s), (B) \rightarrow (r), (C) \rightarrow (q), (D) \rightarrow (p)

$$(A) (16+3-1)C_{3-1} = {}^{18}C_2 = \frac{18 \times 17}{2} = 153$$

$$(B) \frac{n}{2} + 1 = \frac{6}{2} + 1 = 4$$

$$(C) \frac{100}{24} = 4 + F(T_1, T_{25}, T_{49}, T_{73} \text{ are rationals})$$

Total number of irrational terms = $101 - 4 = 97$

$$(D) \text{ put } x = y = 1 \Rightarrow \left(1 + \frac{1}{3} + \frac{2}{3} \right)^{12} = 2^{12}$$

55. (A) $\rightarrow (r, s, t)$, (B) $\rightarrow (s, t)$, (C) $\rightarrow (q, r, s, t)$,
(D) $\rightarrow (p, q, r, s, t)$

We first observe that $n = 8$

(A) $r = 5$

$$(B) \quad 8C_r \cdot 2^r \geq 8C_{r-1} \cdot 2^{r-1} \Leftrightarrow \frac{9-r}{r} \geq \frac{1}{2} \Leftrightarrow r \leq 6$$

$$\text{But for } r = 6, 8C_6 \cdot 2^6 = 8C_5 \cdot 2^5$$

$\therefore T_6$ and T_7 are the terms whose coefficients have greatest value (equal to $7(2^8)$).

(C) $T_{r+1} = nC_r \cdot 2^r x^{r/3}$ is an integer only if $r = 0, 3$ and 6. So the number of integral terms is 3

$$(D) \quad x = \frac{1}{\sqrt{2}} \Rightarrow R = (\sqrt{2} + 1)^8$$

$$= 1 - (\sqrt{2} - 1)^8 + (\sqrt{2} + 1)^8 + (\sqrt{2} - 1)^8 - 1$$

$$= I + f \text{ where } f = 1 - (\sqrt{2} - 1)^8$$

$$\Rightarrow R(1 - f) = (\sqrt{2} + 1)^8 (\sqrt{2} - 1)^8 = 1$$

56. (1) : We have $3^{100} = (3^4)^{25} = (81)^{25} = (80 + 1)^{25}$

$$= {}^{25}C_0 \cdot (80)^{25} + {}^{25}C_1 \cdot (80)^{24} + \dots + {}^{25}C_{22} (80)^3$$

$$+ {}^{25}C_{23} (80)^2 + {}^{25}C_{24} (80) + {}^{25}C_{25}$$

$$= 10^3 [{}^{25}C_0 8^{25} \times 10^{22} + {}^{25}C_1 \times 8^{24} \times 10^{21} + \dots + {}^{25}C_{22} \times 8^3]$$

$$+ \frac{25 \times 24}{2} \times (80)^2 + 25 \times 80 + 1$$

$$= 10^3 m + 1920000 + 2000 + 1, \text{ where } m \in \mathbb{N}$$

$$= 10^3 (m + 1920 + 2) + 1$$

$$\Rightarrow 3^{100} - 1 = 10^3 (m + 1922)$$

$$\Rightarrow 3^{100} - 1 \text{ is divisible by } 1000$$

Thus, last three digits of 3^{100} are 001 i.e., 1

57. (4)

58. (8) : Given, ${}^8C_1 x(1-x)^7 + 2^8 C_2 x^2(1-x)^6$

$$+ 3^8 C_3 x^3(1-x)^5 + \dots + n^8 C_8 x^8 = ax + b$$

Solution we have, ${}^nC_1 x(1-x)^{n-1} + 2^n C_2 x^2(1-x)^{n-2}$

$$+ 3^n C_3 x^3(1-x)^{n-3} + \dots + n^n C_n x^n$$

$$= \sum_{r=1}^n r \cdot {}^nC_r x^r (1-x)^{n-r} = \sum_{r=1}^n r \cdot \frac{n}{r} {}^{n-1}C_{r-1} x^r (1-x)^{n-r}$$

$$= nx \sum_{r=1}^n {}^{n-1}C_{r-1} x^{r-1} (1-x)^{(n-1)-(r-1)}$$

$$= nx [x + (1-x)]^{n-1} = nx = 8x \quad (\because n = 8)$$

$$\Rightarrow a = 8, b = 0 \therefore a + b = 8$$

59. (2)

60. (0) : Coefficient of x^{103} in

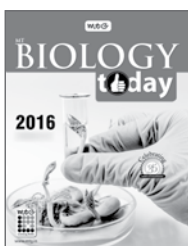
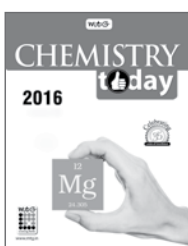
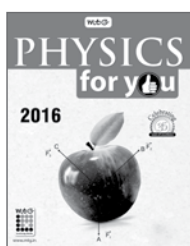
$$(1 + x + x^2 + x^3 + x^4)^{199} (x - 1)^{201}$$

Coefficient of x^{103} in $-(1 - x)^2 (1 - x^5)^{199}$

Coefficient of x^{103} in $-(1 - 2x + x^2)(1 - {}^{199}C_1 x^5 + {}^{199}C_2 x^{10} - {}^{199}C_3 x^{15} + \dots)$

Coefficient of $x^{103} = 0$

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ACE YOUR WAY CBSE

Straight Lines | Conic Sections | Introduction to Three Dimensional Geometry

IMPORTANT FORMULAE

STRAIGHT LINES

- ▶ Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- ▶ Section formula : If the point $P(x, y)$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$, then
 - ▶ $(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$
(Internal Division)
 - ▶ $(x, y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right)$
(External Division)
 - ▶ $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$ (midpoint formula)
- ▶ If $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$ are the vertices and $G(x, y)$, $I(x, y)$ be the centroid and incentre of ΔABC respectively, then
 - ▶ $G(x, y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$
 - ▶ $I(x, y) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$
where a, b, c (respectively) be the sides of the triangle.
- ▶ If $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$, then
 - ▶ Area of $\Delta ABC = \text{mod of } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \text{mod of } \frac{1}{2} \left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} \right]$$
 - ▶ Points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear iff $\text{ar}(\Delta ABC) = 0$
 - ▶ Area of polygon whose vertices are (x_1, y_1) , (x_2, y_2) , ..., (x_n, y_n)

$$= \text{mod of } \frac{1}{2} \left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \dots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix} \right]$$
 - ▶ Slope of a line passing through points (x_1, y_1) and (x_2, y_2) and making an angle θ with positive direction of x-axis is given by $m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$
 - ▶ Let the slopes of two lines be m_1 and m_2 . Then
 - ▶ Lines are parallel iff $m_1 = m_2$
 - ▶ Lines are perpendicular iff $m_1 m_2 = -1$
 - ▶ Angle between two lines $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$
 - ▶ Equation of the line passing through points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

- ▶ Equation of line passing through point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$
- ▶ Equation of line with slope m and y -intercept equal to c is $y = mx + c$
- ▶ Equation of line making intercepts a and b with x and y axes respectively is $\frac{x}{a} + \frac{y}{b} = 1$
- ▶ Length of the perpendicular from the point (α, β) to the line $ax + by + c = 0$ is $\frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}}$
- ▶ The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $d = \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$
- ▶ The lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are said to be concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- ▶ Equation of any line parallel and perpendicular to line $ax + by + c = 0$ is $ax + by + k = 0$ and $bx - ay + k = 0$ respectively.
- ▶ Two points (x_1, y_1) and (x_2, y_2) are on the same side or on opposite side of the line $ax + by + c = 0$ according as the expression $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have same sign or opposite sign.
- ▶ Points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side or opposite side of the line $ax + by + c = 0$ according as this line divides the line segment PQ externally or internally.

CONIC SECTIONS

Circle

- ▶ The equation of a circle with centre (h, k) and radius r is given by $(x - h)^2 + (y - k)^2 = r^2$.
- ▶ The equation of a circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as the end points of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
- ▶ The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$
- ▶ The parametric equations of a circle with centre at (h, k) and radius ' r ' are given by $x = h + r \cos\theta$; $y = k + r \sin\theta$.

Parabola

- ▶ If $y^2 = 4ax$, $a > 0$ is a parabola then,
 - ▶ Focus is $F(a, 0)$ ▶ Vertex is $O(0, 0)$
 - ▶ Equation of directrix is $x + a = 0$
 - ▶ Equation of axis is $y = 0$
 - ▶ Length of latus rectum $= 4a$
 - ▶ Equation of latus rectum is $x = a$.
- ▶ If $x^2 = 4ay$, $a > 0$ is a parabola then,
 - ▶ Focus is $F(0, a)$ ▶ Vertex is $O(0, 0)$
 - ▶ Equation of directrix is $y + a = 0$
 - ▶ Equation of axis is $x = 0$
 - ▶ Length of latus rectum $= 4a$
 - ▶ Equation of latus rectum is $y = a$

Ellipse

- ▶ The standard form of equation of a horizontal ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$.

- ▶ Centre is $O(0, 0)$.
- ▶ Vertices are $A(-a, 0)$ and $B(a, 0)$.
- ▶ Foci are $F_1(-ae, 0)$ and $F_2(ae, 0)$.
- ▶ Length of the major axis $= 2a$ and length of the minor axis $= 2b$.
- ▶ Equation of the major axis is $y = 0$ and that of the minor axis is $x = 0$.
- ▶ Length of the latus rectum $= \frac{2b^2}{a}$
- ▶ Eccentricity, $e = \frac{\sqrt{a^2 - b^2}}{a}$

Hyperbola

- ▶ The standard equation of a horizontal hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
 - ▶ Centre is $O(0, 0)$.
 - ▶ x -axis is the transverse axis and y -axis is the conjugate axis.
 - ▶ Foci are $F_1(-ae, 0)$ and $F_2(ae, 0)$.
 - ▶ Vertices are $A(-a, 0)$ and $B(a, 0)$.
 - ▶ Eccentricity, $e = \frac{\sqrt{a^2 + b^2}}{a}$.
 - ▶ Length of the transverse axis $= 2a$ and its equation is $y = 0$.
 - ▶ Length of the conjugate axis $= 2b$ and its equation is $x = 0$.
 - ▶ Length of its latus rectum $= \frac{2b^2}{a}$.

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

- ▶ **Coordinates of any point**
 - ▶ on x -axis $(x, 0, 0)$, on y -axis $(0, y, 0)$.
 - ▶ on z -axis $(0, 0, z)$.
 - ▶ in xy -plane $(x, y, 0)$.
 - ▶ in yz -plane $(0, y, z)$.
 - ▶ in zx -plane $(x, 0, z)$.
 - ▶ From x -axis $= \sqrt{y^2 + z^2}$
 - ▶ From y -axis $= \sqrt{x^2 + z^2}$
 - ▶ From z -axis $= \sqrt{x^2 + y^2}$
- ▶ If a point moves parallel to x -axis, then its y and z coordinates remain fixed.
- ▶ If a point moves parallel to xy -plane, then its z coordinate remains constant.
- ▶ If $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, then

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

(Distance formula)
- ▶ The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio $m : n$ are given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right) \text{ and } \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right), \text{ respectively.}$$
- ▶ The coordinates of the mid-point of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$
- ▶ The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$
- ▶ If $AB + BC = AC$, then A, B, C are collinear and B lies between A and C .

WORK IT OUT

VERY SHORT ANSWER TYPE

1. If the slope of the line joining $(2, 5)$ and $(3, \lambda)$ is -2 , find the value of λ .
2. Show that the points $A(2, 3, 4)$, $B(-1, 2, -3)$ and $C(-4, 1, -10)$ are collinear. Also, find the ratio in which C divides AB .
3. Find the equation of circle of radius 5, passing through the origin and having its centre on the y -axis.
4. Find the equation of the hyperbola having eccentricity $e = \frac{4}{3}$ and vertices at $(0, \pm 7)$.
5. Find the equation of the parabola whose focus is $(1, 1)$ and the directrix $x + y + 1 = 0$.

SHORT ANSWER TYPE

6. Determine the point in XY -plane which is equidistant from the three points $(2, 0, 3)$, $(0, 3, 2)$ and $(0, 0, 1)$.
7. Find the points of trisection of the line segment joining the points $(2, -2, 7)$ and $(5, 1, -5)$.
8. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h .

9. The girder of a railway bridge is a parabola with its vertex at the highest point, 10 m above the ends. If the span is 100 m, find its height at 20 m from the mid-point.
10. Find the equation of the circle passing through the point $(2, 4)$ and centre at the intersection of the lines $x - y = 4$ and $2x + 3y = -7$.

LONG ANSWER TYPE - I

11. The foci of a hyperbola coincides with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2.
12. Show that the points $(5, 5)$, $(6, 4)$, $(-2, 4)$ and $(7, 1)$ all lie on a circle and find its equation, centre and radius.
13. If for a variable line $\frac{x}{a} + \frac{y}{b} = 1$, the condition $a^{-2} + b^{-2} = c^{-2}$ (c is a constant) is satisfied, find the locus of foot of the perpendicular drawn from the origin to this line.
14. The vertices of a triangle are $A(5, 4, 6)$, $B(1, -1, 3)$ and $C(4, 3, 2)$. The internal bisector of $\angle BAC$ meets BC in D . Find AD .
15. Find the equation of the ellipse that passes through the origin and has the foci at the points $(-1, 1)$ and $(1, 1)$.

LONG ANSWER TYPE - II

16. Find the equation of the hyperbola whose foci are at $(0, \pm\sqrt{10})$ and which passes through the point $(2, 3)$.
17. The vertices of a triangle are $A(10, 4)$, $B(-4, 9)$ and $C(-2, -1)$. Find the equation of the altitude through A . Also, find the foot of this altitude.
18. Find the equation of the line midway between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.
19. Find the lengths and equations of major and minor axes, centre, eccentricity, foci, equation of directrices, vertices and length of latus rectums for the ellipse $16x^2 + y^2 = 16$.
20. If y_1, y_2, y_3 be the ordinates of vertices of a triangle inscribed in a parabola $y^2 = 4ax$, then show that the area of the triangle is $\left| \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \right|$.

SOLUTIONS

1. Let $A = (2, 5)$ and $B = (3, \lambda)$

$$\text{Slope of } AB = -2 \Rightarrow -2 = \frac{\lambda - 5}{3 - 2}$$

$$\Rightarrow -2 = \lambda - 5 \Rightarrow \lambda = 5 - 2 = 3$$

2. Given, $A \equiv (2, 3, 4)$, $B \equiv (-1, 2, -3)$, $C \equiv (-4, 1, -10)$
Let C divide AB in the ratio $k : 1$, then

$$C \equiv \left(\frac{-k + 2}{k + 1}, \frac{2k + 3}{k + 1}, \frac{-3k + 4}{k + 1} \right)$$

$$\therefore \frac{-k + 2}{k + 1} = -4 \Rightarrow 3k = -6 \Rightarrow k = -2$$

$$\text{For this value of } k, \frac{2k + 3}{k + 1} = 1 \text{ and } \frac{-3k + 4}{k + 1} = -10$$

Hence, C divides AB externally in the ratio $2 : 1$.

3. The radius of circle = 5.
The circle passes through the origin and its centre is on the y -axis.

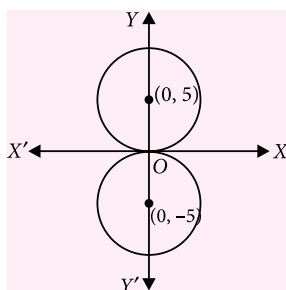
Equation of circles are

$$(x - 0)^2 + (y - 5)^2 = 5^2$$

$$\Rightarrow x^2 + y^2 - 10y = 0$$

$$\text{and } (x - 0)^2 + (y + 5)^2 = 5^2$$

$$\Rightarrow x^2 + y^2 + 10y + 25 = 25 \text{ or } x^2 + y^2 + 10y = 0$$



4. The vertices of the hyperbola are at $(0, \pm 7)$. These are on the y -axis. Centre of the hyperbola will be $(0, 0)$.

Let the equation of the hyperbola be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$... (i)

where $a, b > 0$ and $b^2 = a^2(e^2 - 1)$

The vertices of this hyperbola are $(0, \pm a) \therefore a = 7$

$$\text{Now, } b^2 = a^2(e^2 - 1) = 7^2 \left[\left(\frac{4}{3} \right)^2 - 1 \right] = \frac{343}{9}$$

\therefore From (i), we get

$$\text{Required equation} = \frac{y^2}{(7)^2} - \frac{x^2}{\frac{343}{9}} = 1 \text{ or } \frac{y^2}{49} - \frac{9x^2}{343} = 1$$

5. Let $P(x, y)$ be any point on the parabola.

Then the distance of $P(x, y)$ from the focus $(1, 1)$

= distance of $P(x, y)$ from the directrix $x + y + 1 = 0$

$$\therefore PS = PN$$

$$\Rightarrow \sqrt{(x - 1)^2 + (y - 1)^2} = \frac{|x + y + 1|}{\sqrt{(1)^2 + (1)^2}} \quad \dots (i)$$

Squaring (i), we get

$$(x - 1)^2 + (y - 1)^2 = \left(\frac{x + y + 1}{\sqrt{2}} \right)^2$$

$$\Rightarrow 2[x^2 + 1 - 2x + y^2 + 1 - 2y]$$

$$= x^2 + y^2 + 2xy + 2y + 2x + 1$$

$$\Rightarrow x^2 - 2xy + y^2 - 6x - 6y + 3 = 0$$

6. Let $P(x, y, 0)$ be the point in XY -plane which is equidistant from $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.

Then, $AP = BP = CP \Rightarrow AP^2 = BP^2 = CP^2$.

Now, $AP^2 = CP^2$

$$\Rightarrow (x - 2)^2 + (y - 0)^2 + (0 - 3)^2 = (x - 0)^2 + (y - 0)^2 + (0 - 1)^2$$

$$\Rightarrow x^2 + 4 - 4x + y^2 + 9 = x^2 + y^2 + 1$$

$$\Rightarrow -4x = -12 \Rightarrow x = 3.$$

Again, $BP^2 = CP^2$

$$\Rightarrow (x - 0)^2 + (y - 3)^2 + (0 - 2)^2 = (x - 0)^2 + (y - 0)^2 + (0 - 1)^2$$

$$\Rightarrow x^2 + y^2 + 9 - 6y + 4 = x^2 + y^2 + 1 \Rightarrow y = 2$$

Hence the required point is $P(x, y, 0) = P(3, 2, 0)$.

7. Let P and Q be the points of trisection.

Then, $AP = PQ = QB$

$\therefore P$ divides AB internally in the ratio $1 : 2$

$$\therefore P \equiv \left(\frac{1 \times 5 + 2 \times 2}{2 + 1}, \frac{1 \times 1 + 2 \times (-2)}{2 + 1}, \frac{1 \times (-5) + 2 \times 7}{2 + 1} \right)$$

or $P \equiv (3, -1, 3)$. Now, Q is the mid-point of PB

$$\therefore Q \equiv \left(\frac{3+5}{2}, \frac{-1+1}{2}, \frac{3-5}{2} \right) \equiv (4, 0, -1)$$

8. Slope of the line PQ passing through $P(h, 3)$ and $Q(4, 1)$

$$\text{is } \frac{3-1}{h-4} = \frac{2}{h-4}$$

Slope of the line AB ,

$$7x - 9y - 19 = 0 \text{ is } \frac{7}{9}.$$

The lines AB and PQ are perpendicular to each other.

$$\therefore \frac{2}{h-4} \times \frac{7}{9} = -1$$

$$\Rightarrow 14 = -9(h-4) \Rightarrow 9h = 36 - 14 = 22 \Rightarrow h = \frac{22}{9}$$

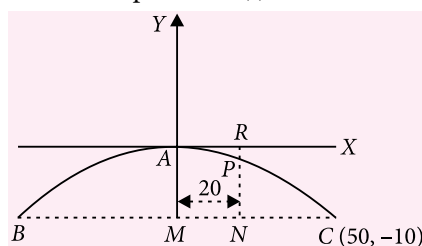
9. The girder BAC of the bridge has been shown in the figure. We take the vertex A of the parabola as origin and the axis of the parabola as y -axis. Then equation of the parabola will be $x^2 = -4ay$, $a > 0$... (i)

Given, $AM = 10$ m and $BC = 100$ m

As M is the mid-point of BC

$\therefore MC = 50$ m and $C \equiv (50, -10)$

Since C lies on the parabola (i)



$$\therefore 50^2 = -4a \cdot (-10) \Rightarrow a = \frac{250}{4}$$

Let $MN = 20$ m, we draw $NR \perp AX$ to meet the parabola in P .

As P lies below x -axis, coordinates of P are $(20, -\beta)$, where $\beta = PR > 0$.

Since P lies on the parabola (i)

$$\therefore 20^2 = -4 \cdot \frac{250}{4} \cdot (-\beta) \Rightarrow \beta = \frac{400}{250} = \frac{8}{5} = 1.6$$

\therefore Required height $= NR - PR = 10 - 1.6 = 8.4$ m.

10. The point of intersection of the lines $x - y - 4 = 0$ and $2x + 3y + 7 = 0$ is $(1, -3)$.

Now, centre of circle is $(1, -3)$, and it passes through the point $(2, 4)$.

$$\therefore \text{Radius} = \sqrt{(2-1)^2 + (4+3)^2} = \sqrt{1+49} = \sqrt{50}$$

Hence the required equation of the circle is

$$(x-1)^2 + (y+3)^2 = (\sqrt{50})^2 \Rightarrow x^2 + y^2 - 2x + 6y - 40 = 0$$

11. The given ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$ or $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$

Here $5 > 3 \therefore a = 5$ and $b = 3$

$$\therefore \text{Eccentricity of the ellipse, } e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

\therefore Foci of the ellipse are $(\pm ae, 0)$ or $(\pm 5(4/5), 0)$ or $(\pm 4, 0)$

Also centre of the hyperbola will be $(0, 0)$.

Let the equation of the hyperbola be $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$

\therefore Foci are $(\pm Ae, 0)$ or $(\pm 2A, 0)$ [$\because e = 2$]

$\therefore 2A = 4 \Rightarrow A = 2$ and

$$B = A\sqrt{e^2 - 1} = 2\sqrt{(2)^2 - 1} = 2\sqrt{3}$$

\therefore The equation of the hyperbola is

$$\frac{x^2}{(2)^2} - \frac{y^2}{(2\sqrt{3})^2} = 1 \text{ or } \frac{x^2}{4} - \frac{y^2}{12} = 1$$

12. Let the equation of the circle through $(5, 5)$, $(6, 4)$ and $(-2, 4)$ be $x^2 + y^2 + 2gx + 2fy + c = 0$... (i)

As (i) passes through $(5, 5)$, $(6, 4)$ and $(-2, 4)$, we get

$$25 + 25 + 10g + 10f + c = 0$$

$$\text{or } 10g + 10f + c = -50$$

... (ii)

$$36 + 16 + 12g + 8f + c = 0$$

$$\text{or } 12g + 8f + c = -52$$

... (iii)

$$4 + 16 - 4g + 8f + c = 0 \text{ or } -4g + 8f + c = -20$$

... (iv)

Subtracting (ii) from (iii) and (iv), we get

$$2g - 2f = -2 \text{ or } g - f = -1$$

... (v)

$$\text{and } -14g - 2f = 30 \text{ or } 7g + f = -15$$

... (vi)

Adding (v) and (vi), we get $8g = -16 \Rightarrow g = -2$

Putting $g = -2$ in (v), we get $f = -2 + 1 = -1$

Putting $g = -2$, $f = -1$ in (ii), we get

$$10(-2) + 10(-1) + c = -50 \text{ or } c = -20$$

Putting the values of g , f and c in (i), we get

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

... (vii)

To check if $(7, 1)$ lies on (vii), we put $x = 7$ and

$y = 1$ in (vii)

$$7^2 + 1^2 - 4(7) - 2(1) - 20 = 0$$

$$\text{or } 49 + 1 - 28 - 2 - 20 = 0 \text{ or } 0 = 0.$$

Thus, $(7, 1)$ lies on (vii), Centre of (vii) is $(2, 1)$ and

$$\text{radius is } \sqrt{2^2 + 1^2 - (-20)} = \sqrt{25} = 5.$$

13. The given line is $\frac{x}{a} + \frac{y}{b} = 1$... (i)

Equation of the line \perp to (i) from $(0, 0)$ is $\frac{x}{b} - \frac{y}{a} = 0$... (ii)

Let (α, β) be the point of intersection of (i) and (ii).

Then $\frac{\alpha}{a} + \frac{\beta}{b} = 1$... (iii) and $\frac{\alpha}{b} - \frac{\beta}{a} = 0$... (iv)

Squaring and adding (iii) and (iv), we get

$$\alpha^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \beta^2 \left(\frac{1}{b^2} + \frac{1}{a^2} \right) = 1$$

$$\Rightarrow (\alpha^2 + \beta^2) \left(\frac{1}{a^2} + \frac{1}{b^2} \right) = 1 \Rightarrow (\alpha^2 + \beta^2) \left(\frac{1}{c^2} \right) = 1$$

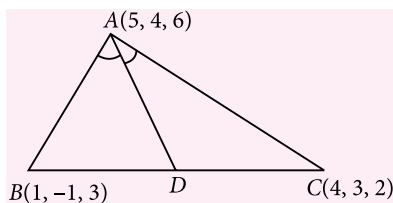
$$\Rightarrow \alpha^2 + \beta^2 = c^2.$$

Thus, locus of (α, β) is $x^2 + y^2 = c^2$.

14. $AB = \sqrt{4^2 + 5^2 + 3^2} = 5\sqrt{2}$

$$AC = \sqrt{1^2 + 1^2 + 4^2} = 3\sqrt{2}$$

Since AD is the internal bisector of $\angle BAC$



$$\therefore \frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3}$$

Also, D divides BC internally in the ratio $5 : 3$

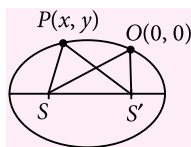
$$\therefore D \equiv \left(\frac{5 \times 4 + 3 \times 1}{5 + 3}, \frac{5 \times 3 + 3 \times (-1)}{5 + 3}, \frac{5 \times 2 + 3 \times 3}{5 + 3} \right) \\ \equiv \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8} \right)$$

$$\therefore AD = \sqrt{\left(5 - \frac{23}{8} \right)^2 + \left(4 - \frac{12}{8} \right)^2 + \left(6 - \frac{19}{8} \right)^2} \\ = \sqrt{\frac{17^2 + 20^2 + 29^2}{8^2}} = \frac{\sqrt{1530}}{8} \text{ units}$$

15. Let $P(x, y)$ be any point on the ellipse and the foci be $S(-1, 1)$ and $S'(1, 1)$.

Now, P and O lies on the ellipse

$$\Rightarrow PS + PS' = \text{constant} = OS + OS'$$



[Since sum of focal distances of any point on the ellipse is constant.]

$$\Rightarrow \sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2\sqrt{2}$$

$$\Rightarrow \sqrt{(x+1)^2 + (y-1)^2} = 2\sqrt{2} - \sqrt{(x-1)^2 + (y-1)^2}$$

$$\Rightarrow (x+1)^2 + (y-1)^2 = 8 + [(x-1)^2 + (y-1)^2] - 4\sqrt{2} \sqrt{(x-1)^2 + (y-1)^2}$$

$$\Rightarrow x^2 + 2x + 1 + y^2 - 2y + 1 = 8 + x^2 - 2x + 1 + y^2 - 2y + 1 - 4\sqrt{2} \sqrt{(x-1)^2 + (y-1)^2}$$

$$\Rightarrow 4x - 8 = -4\sqrt{2} \sqrt{(x-1)^2 + (y-1)^2}$$

$$\Rightarrow x - 2 = -\sqrt{2} \sqrt{(x-1)^2 + (y-1)^2}$$

$$\Rightarrow x^2 - 4x + 4 = 2[x^2 - 2x + 1 + y^2 - 2y + 1]$$

$\Rightarrow x^2 + 2y^2 - 4y = 0$ is the required equation of the ellipse.

16. Since the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$... (i)

Let its foci be $(0, \pm c)$.

But, the foci are $(0, \pm\sqrt{10})$.

$$\therefore c = \sqrt{10} \Leftrightarrow c^2 = 10 \Leftrightarrow a^2 + b^2 = 10 \quad \dots (ii)$$

$$[\because c^2 = a^2 + b^2]$$

Since (i) passes through $(2, 3)$, we have $\frac{9}{a^2} - \frac{4}{b^2} = 1$.

$$\text{Now, } \frac{9}{a^2} - \frac{4}{b^2} = 1 \Leftrightarrow \frac{9}{a^2} - \frac{4}{(10-a^2)} = 1 \quad [\text{Using (ii)}]$$

$$\Leftrightarrow 9(10-a^2) - 4a^2 = a^2(10-a^2)$$

$$\Leftrightarrow a^4 - 23a^2 + 90 = 0 \Leftrightarrow (a^2 - 18)(a^2 - 5) = 0$$

$$\Leftrightarrow a^2 = 5$$

$$[\because a^2 = 18 \Rightarrow b^2 = -8, \text{ which is not possible}]$$

Thus, $a^2 = 5$ and $b^2 = 5$.

Hence, the required equation is $\frac{y^2}{5} - \frac{x^2}{5} = 1$

i.e., $y^2 - x^2 = 5$.

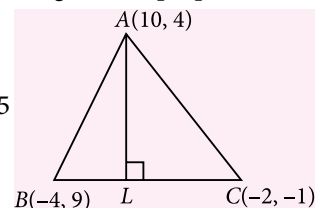
17. Altitude AL is the line through A and perpendicular to BC as shown.

Now, slope of

$$BC = \frac{9 - (-1)}{-4 - (-2)} = \frac{10}{-2} = -5$$

$$\therefore \text{Slope of } AL = 1/5$$

Equation of the altitude



$$\begin{aligned}
 AL \text{ is } y - 4 &= \frac{1}{5}(x - 10) \\
 \Rightarrow 5y - 20 &= x - 10 \\
 \Rightarrow x - 5y + 10 &= 0 \quad \dots(i)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, equation of } BC \text{ is } y - 9 &= \frac{-1 - 9}{-2 + 4}(x + 4) \\
 \Rightarrow y - 9 &= -5(x + 4) \text{ or } 5x + y + 11 = 0 \quad \dots(ii)
 \end{aligned}$$

Now, L is the point of intersection of BC and AL . Solving (i) and (ii) simultaneously for x and y , we get

$$\frac{x}{10 + 55} = \frac{y}{11 - 50} = \frac{1}{-25 - 1} \Rightarrow x = -\frac{5}{2}, y = \frac{3}{2}.$$

$$\therefore \text{Foot of perpendicular, } L \equiv \left(-\frac{5}{2}, \frac{3}{2}\right).$$

18. Converting each of the given equations to the form $y = mx + C$, we get

$$9x + 6y - 7 = 0 \Rightarrow y = \frac{-3}{2}x + \frac{7}{6} \quad \dots(i)$$

$$3x + 2y + 6 = 0 \Rightarrow y = \frac{-3}{2}x - 3 \quad \dots(ii)$$

Clearly, the slope of each one of the given lines is $-\frac{3}{2}$.

Let the given lines be $y = mx + C_1$ and $y = mx + C_2$.

$$\text{Then, } m = \frac{-3}{2}, C_1 = \frac{7}{6} \text{ and } C_2 = -3.$$

Let L be the required line. Then, L is parallel to each one of (i) and (ii), and equidistant from each one of them.

$$\therefore \text{Slope of } L = \frac{-3}{2}.$$

$$\text{Let the equation of } L \text{ be } y = \frac{-3}{2}x + C \quad \dots(iii)$$

Then, distance between (i) and (iii) must be equal to the distance between (ii) and (iii).

$$\therefore \frac{|C_1 - C|}{\sqrt{1 + m^2}} = \frac{|C_2 - C|}{\sqrt{1 + m^2}} \Rightarrow |C_1 - C| = |C_2 - C|$$

$$\Rightarrow \left|\frac{7}{6} - C\right| = |-3 - C| \Rightarrow \left|\frac{7}{6} - C\right| = |3 + C|$$

$$\Rightarrow \frac{7}{6} - C = 3 + C \Rightarrow 2C = \frac{-11}{6} \Rightarrow C = \frac{-11}{12}.$$

$$\therefore \text{Equation of } L \text{ is } y = \frac{-3}{2}x - \frac{11}{12}$$

i.e., $18x + 12y + 11 = 0$.

Hence, the line $18x + 12y + 11 = 0$ is midway between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

19. Given equation of the ellipse is

$$16x^2 + y^2 = 16 \text{ or } \frac{x^2}{1^2} + \frac{y^2}{4^2} = 1 \quad \dots(i)$$

Here $4 > 1 \therefore a = 4, b = 1$

Length of major axis $= 2a = 8$

Length of minor axis $= 2b = 2$

Equation of major axis is $x = 0$

Equation of minor axis is $y = 0$

Coordinates of centre are $(0, 0)$

$$\text{Eccentricity of the ellipse } e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$$

Coordinates of foci are given by $y = \pm ae, x = 0$

$$\text{or } y = \pm \sqrt{15}, x = 0$$

Hence foci are $(0, \pm \sqrt{15})$.

$$\text{Equation of directrices are } y = \pm \frac{a}{e} \text{ or } y = \pm \frac{16}{\sqrt{15}}$$

Coordinates of vertices are given by $(0, \pm 4)$.

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{2 \times 1}{4} = \frac{1}{2}.$$

20. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$. Since (x_1, y_1) , (x_2, y_2) and (x_3, y_3) lie on the parabola, $y^2 = 4ax$

$$\therefore y_1^2 = 4ax_1, y_2^2 = 4ax_2 \text{ and } y_3^2 = 4ax_3$$

$$\Rightarrow x_1 = \frac{y_1^2}{4a}, x_2 = \frac{y_2^2}{4a} \text{ and } x_3 = \frac{y_3^2}{4a}$$

Now, area of $\triangle ABC$

$$= \frac{1}{2} |(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))|$$

$$= \frac{1}{2} \left| \frac{y_1^2}{4a}(y_2 - y_3) + \frac{y_2^2}{4a}(y_3 - y_1) + \frac{y_3^2}{4a}(y_1 - y_2) \right|$$

$$= \frac{1}{8|a|} |(y_2 - y_3)[y_1^2 + y_2y_3 - y_1(y_2 + y_3)]|$$

$$= \frac{1}{8|a|} |(y_2 - y_3)[y_1(y_1 - y_2) - y_3(y_1 - y_2)]|$$

$$= \left| \frac{1}{8|a|} (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \right|$$

$$\text{Hence, area of } \triangle ABC = \left| \frac{1}{8a} (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \right|$$

Solution Sender of Maths Musing

SET-177

- Satya Dev (Bangalore)
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SET-178

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MPP-7 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Limits and Derivatives

Total Marks : 80

Time Taken : 60 Min.

Only One Option Correct Type

- The value of $\lim_{x \rightarrow 1/2} \left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1} \right)$ is
(a) $\frac{7}{3}$ (b) $-\frac{7}{2}$ (c) $-\frac{7}{3}$ (d) $\frac{7}{2}$
- If $f(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to
(a) $\frac{1}{100}$ (b) 100 (c) 1 (d) 0
- $\lim_{n \rightarrow \infty} \left(\frac{1}{5} \right)^{\log_{\sqrt{5}} \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^{n+1}} \right)}$ equals
(a) 2 (b) 4 (c) 8 (d) 0
- The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$, is
(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) 1 (d) -1
- $\lim_{x \rightarrow \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$ is equal to
(a) 3 (b) -2 (c) 2 (d) 4
- $\lim_{n \rightarrow \infty} \left(\frac{n!}{(mn)^n} \right)^{\frac{1}{n}}$ equals
(a) em (b) $\frac{1}{em}$ (c) $\frac{m}{e}$ (d) $\frac{e}{m}$

One or More Than One Option(s) Correct Type

- Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$, then $\frac{dy}{dx}$ is equal to
(a) $\frac{1}{2y-1}$ (b) $\frac{y}{x+y^2}$
(c) $\frac{1}{\sqrt{1+2x}}$ (d) $\frac{y}{2x+y}$
- If $\lim_{x \rightarrow 1} (2 - x + a[x-1] + b[1+x])$ exists, then find the possible values of a and b (where $[.]$ denotes the greatest integer function).
(a) $a = 1/3, b = 1$ (b) $a = 1, b = -1$
(c) $a = 9, b = -9$ (d) $a = 2, b = 2/3$
- Let $f(x) = \begin{cases} 1 + \frac{2x}{a}, & 0 \leq x < 1 \\ ax, & 1 \leq x < 2 \end{cases}$. If $\lim_{x \rightarrow 1} f(x)$ exists, then a is
(a) 1 (b) -1 (c) 2 (d) -2

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10. If $f(x) = \sqrt{1 + \cos^2(x)^2}$, then $f'(\sqrt{\pi})$ is

- (a) $\frac{\sqrt{\pi}}{6}$ (b) 0 (c) $\frac{1}{\sqrt{6}}$ (d) $\frac{\pi}{\sqrt{6}}$

11. $f(x) = |x^2 - 3|x| + 2|$. Then which of the following is/are true?

- (a) $f'(x) = 2x - 3$ for $x \in (0, 1) \cup (2, \infty)$
 (b) $f'(x) = 2x + 3$ for $x \in (-\infty, -2) \cup (-1, 0)$
 (c) $f'(x) = -2x - 3$ for $x \in (-2, -1)$
 (d) None of these

12. Given a real-valued function f such that

$$f(x) = \begin{cases} \frac{\tan^2\{x\}}{(x^2 - [x]^2)}, & \text{for } x > 0 \\ 1, & \text{for } x = 0 \\ \sqrt{\{x\} \cot\{x\}}, & \text{for } x < 0 \end{cases}$$

where $[x]$ is the integral part and $\{x\}$ is the fractional part of x , then

- (a) $\lim_{x \rightarrow 0^+} f(x) = 1$ (b) $\lim_{x \rightarrow 0^-} f(x) = \cot 1$

(c) $\left(\lim_{x \rightarrow 0^-} f(x) \right)^2 = \cot 1$

(d) None of these

13. If $y = \sqrt{\frac{1-x}{1+x}}$, then $(1-x^2) \frac{dy}{dx}$ is equal to

- (a) y^2 (b) $\frac{1}{y}$ (c) $-y$ (d) $-\frac{y}{x}$

Comprehension Type

If $L = \lim_{x \rightarrow 0} \frac{\sin x + ae^x + be^{-x} + c \ln(1+x)}{x^3}$ exist finitely, then

14. The value of L is

- (a) $1/2$ (b) $-1/3$ (c) $-1/6$ (d) 3

15. The solution set of $||x + c| - 2a| < 4b$ is

- (a) $[-2, 2]$ (b) $[0, 2]$
 (c) $(-1, 1)$ (d) $[-2, 1]$

Matrix Match Type

16. Match the following.

Column I		Column II	
P.	If $y = \frac{1}{x}$, then $\frac{dy}{dx} \sqrt{\frac{1+x^4}{1+y^4}}$ is	1.	8
Q.	For the function $f(x) = \ln \tan x $, $f'\left(-\frac{\pi}{4}\right)$ is equal to	2.	1
R.	If $f(x) = (x-1)(x-2) \dots (x-n)$, $n \in N$ and $f'(n) = 5040$, then n is	3.	-1
S.	The derivatives of $\frac{\log x }{x}$ at $x = -1$ is	4.	2

	P	Q	R	S
(a)	3	4	1	2
(b)	4	3	2	1
(c)	1	2	4	3
(d)	3	1	4	2

Integer Answer Type

17. The reciprocal of the value of

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{n^2}\right) \text{ is}$$

18. Let $g(x) = \begin{cases} \frac{x^2 + x \tan x - x \tan 2x}{ax + \tan x - \tan 3x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

If $g'(0)$ exists and is equal to nonzero value b , then $52 \frac{b}{a}$ is equal to

19. The value of $\lim_{x \rightarrow \infty} \frac{\log_e(\log_e x)}{e^{\sqrt{x}}}$ is

20. If $f(x) = \sqrt{x + 2\sqrt{2x - 4}} + \sqrt{x - 2\sqrt{2x - 4}}$, then the value of $10 f'(102^+)$ is



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SELF CHECK

No. of questions attempted
 No. of questions correct
 Marks scored in percentage

Check your score! If your score is

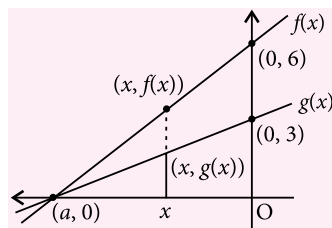
> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY !	Revise thoroughly and strengthen your concepts.

BRAIN @ WORK



LIMITS, CONTINUITY AND DIFFERENTIABILITY, APPLICATION OF DERIVATIVES

1. Suppose you have two linear functions f and g as shown in the given figure.



Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is

- (a) does not exist
(b) not enough information
(c) 2 (d) 3
2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(1) = 3$ and $f'(1) = 6$. Then, $\lim_{x \rightarrow 0} \left\{ \frac{f(1+x)}{f(1)} \right\}^{1/x}$ is equal to
(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3
3. $\lim_{x \rightarrow \infty} \left\{ \frac{1^2}{1-x^3} + \frac{3}{1+x^2} + \frac{5^2}{1-x^3} + \frac{7}{1+x^2} + \dots \right\} =$
(a) $-5/6$ (b) $-10/3$ (c) $5/6$ (d) $21/3$
4. $\lim_{x \rightarrow 0} \frac{(2^{\sin x} - 1)[\ln(1 + \sin 2x)]}{x \tan^{-1} x}$ is equal to
(a) $\ln 2$ (b) $2 \ln 2$ (c) $(\ln 2)^2$ (d) 0
5. If $f(x) = \begin{cases} \frac{[x]^2 + \sin[x]}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$
where $[x]$ denotes the greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$ equals
(a) 1 (b) 0
(c) -1 (d) none of these
6. If $\lim_{n \rightarrow \infty} \frac{5^{n+1} + 3^n - 3^{2n}}{5^n + 2^n + 3^{2n+3}}$ is equal to
(a) 5 (b) 3
(c) 1 (d) none of these
7. If f is an even function such that $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$ has some finite non-zero value, then
(a) f is continuous and derivable at $x = 0$
(b) f is continuous but not derivable at $x = 0$
(c) f may be discontinuous at $x = 0$
(d) none of these
8. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0 (d) all integers except 1
9. If $f(x) = \frac{1}{1-x}$, then the points of discontinuity of the function $f^{3n}(x)$ is/are, where $f^n = f \circ f \dots \circ f$ (n times), are
(a) $x = 2$ (b) $x = \{0, 1\}$
(c) $x = -1$ (d) none of these
10. If $f(x) = \begin{cases} (1 + |\sin x|)^{a/|\sin x|}, & -\pi/6 < x < 0 \\ b, & x = 0 \\ e^{\tan 2x / \tan 3x}, & 0 < x < \pi/6 \end{cases}$ is a continuous function on $(-\pi/6, \pi/6)$, then
(a) $a = 2/3, b = e^2$ (b) $a = 1/3, b = e^{1/3}$
(c) $a = 2/3, b = e^{2/3}$ (d) none of these
11. Let $f(x) = \begin{cases} [x], & \text{if } x \notin I \\ x-1, & \text{if } x \in I \end{cases}$ (where, $[\cdot]$ denotes the greatest integer function),
 $g(x) = \begin{cases} \sin x + \cos x, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$. Then

- (a) $\lim_{x \rightarrow 0} f(g(x))$ exist but $f(g(x))$ is not continuous
 (b) $f(g(x))$ is continuous but not differentiable at $x = 0$
 (c) $f(g(x))$ is differentiable at $x = 0$
 (d) $\lim_{x \rightarrow 0} f(g(x))$ does not exist
12. Which of the following function is discontinuous in R ?
 (a) $\sin^2 x + \cos^2 x$ (b) $x^2|x| + \sin x$
 (c) $\frac{1}{x^2 + 100}$ (d) $\frac{1}{x^2 - 1}$
13. Function $f(x) = \frac{\tan \pi[x - \pi]}{1 + [x]^2}$ (where $[.]$ is GIF) is
 (a) continuous at integers only
 (b) continuous in R^+ only
 (c) continuous in R^- only
 (d) Everywhere continuous
14. A function is defined as $f(x) = \begin{cases} e^x, & x \leq 0 \\ |x-1|, & x > 0 \end{cases}$
 then $f(x)$ is
 (a) continuous at $x = 0, 1$
 (b) differentiable at $x = 0$
 (c) differentiable at $x = 1$
 (d) none of these
15. If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $f(x)$ is differentiable on
 (a) $[-1, 1]$ (b) $R - \{-1, 1\}$
 (c) $R - (-1, 1)$ (d) none of these
16. If $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ |x+3|, & x < 1 \end{cases}$ then the function is not differentiable at
 (a) one point (b) two points
 (c) three point (d) four points
17. P is a variable point on the curve $y = f(x)$ and A is a fixed point in the plane not lying on the curve. If PA^2 is minimum, then the angle between PA and the tangent at P is
 (a) $\pi/4$ (b) $\pi/3$
 (c) $\pi/2$ (d) none of these
18. The points of contact of the vertical tangents to $x = 2 - 3 \sin \theta, y = 3 + 2 \cos \theta$ are
 (a) $(2, 5), (2, 1)$ (b) $(-1, 3), (5, 3)$
 (c) $(2, 5), (5, 3)$ (d) $(-1, 3), (2, 1)$
19. The angle of intersection of curves $y = [\sin x] + [\cos x]$ and $x^2 + y^2 = 5$ where $[.]$ denotes the greatest integer function, is
 (a) $\tan^{-1}(2)$ (b) $\tan^{-1}(1/2)$
 (c) $\tan^{-1}(\sqrt{2})$ (d) $\pi/2$
20. Let $f(x)$ is differentiable function in $[0, 2]$. $f(0) = 0$ and $f'(x) \leq \frac{1}{2} \forall x \in [0, 2]$, then
 (a) $|f(x)| \leq 2$ (b) $f(x) \leq 1$
 (c) $f(x) = 2x$
 (d) $f(x) = 3$ for some $x \in (0, 2)$
21. The equation of tangent to the graph of the function $f(x) = |x^2 - |x||$ at the point with abscissa $x = -2$ is
 (a) $3x + y + 4 = 0$ (b) $5x - y - 12 = 0$
 (c) $5x + y + 8 = 0$ (d) $3x + y - 4 = 0$
22. A kite is 100 ft. high and there are 260 ft. of cord out. If the kite is moving horizontally at rate of $6\frac{1}{2}$ miles per hour directly away from the person who is flying it; how fast is the cord being paid out?
 (a) $\frac{6}{5}$ miles/hour (b) 6 miles/hour
 (c) $6\frac{1}{2}$ miles/hour (d) 3 miles/hour
23. If $f(x) = x(x-2)(x-4)$, $1 \leq x \leq 4$, then a number satisfying the condition of the mean value theorem is
 (a) 1 (b) 2 (c) $5/2$ (d) $7/2$
24. If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one real root lying between
 (a) 0 and 1 (b) 1 and 3
 (c) 0 and 3 (d) none of these
25. The value of 'a' for which the equation $x^3 - 3x + a = 0$ has two different roots in $[0, 1]$ is given by
 (a) -1 (b) 2
 (c) 1 (d) none of these
26. The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ drawn at the point $x = 0$ is
 (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$
27. If $f(x)$ and $g(x) = f(x)\sqrt{1 - 2(f(x))^2}$ are monotoni-cally increasing, then $\forall x \in R$
 (a) $|f(x)| \leq 1$ (b) $|f(x)| < \frac{2}{3}$
 (c) $|f(x)| < \frac{1}{2}$ (d) $|f(x)| < \frac{1}{\sqrt{2}}$
28. Let $f(x) = 2x^3 + ax^2 + bx - 3\cos^2 x$ is an increasing function for all $x \in R$, then
 (a) $a^2 - 6b - 18 > 0$ (b) $a^2 - 6b + 18 < 0$
 (c) $a^2 - 3b - 6 < 0$ (d) $a > 0, b > 0$

29. If the function $f(x) = \cos|x| - 2ax + b$ increases along the entire number scale, the range of values of a is given by
 (a) $a \leq b$ (b) $a = 1/2$
 (c) $a \leq -1/2$ (d) $a \geq -3/2$
30. If $\phi(x) = 3f\left(\frac{x^2}{3}\right) + f(3-x^2) \forall x \in (-3, 4)$ where $f''(x) > 0 \forall x \in (-3, 4)$, then $\phi(x)$ is
 (a) increasing in $\left(\frac{3}{2}, 4\right)$
 (b) increasing in $\left(-3, -\frac{3}{2}\right)$
 (c) decreasing in $\left(-\frac{3}{2}, 0\right)$
 (d) increasing in $\left(0, \frac{3}{2}\right)$
31. The function $f(x) = 1 + x(\sin x) [\cos x]$, $0 < x \leq \pi/2$
 (a) is continuous on $(0, \pi/2)$
 (b) is strictly decreasing in $(0, \pi/2)$
 (c) is strictly increasing in $(0, \pi/2)$
 (d) has global maximum value 2
32. If M be the greatest value and m be the least value of $f(x) = 2x^3 - 3x^2 - 12x + 1$, for $-1 \leq x \leq 3/2$, then the ordered pair (M, m) is
 (a) $(8, -19)$ (b) $(8, -17)$
 (c) $(-17, -19)$ (d) none of these
33. $f(x) = x^9 + 3x^7 + 64$ is monotonic increasing for
 (a) positive real values of x
 (b) negative real values of x
 (c) all real values of x
 (d) all non zero values of x
34. If for a function f , $|f(x) - f(y)| \leq (x - y)^2 \forall x, y \in R$, then f is
 (a) strictly increasing (b) strictly decreasing
 (c) constant
 (d) strictly increasing when $x > 0$
35. If the function $f(x) = 2 \tan x + (2a + 1) \log_e |\sec x| + (a - 2)x$ is increasing on R , then
 (a) $a \in (1/2, \infty)$ (b) $a \in (-1/2, 1/2)$
 (c) $a = 1/2$ (d) $a \in R$
36. If f and g are two increasing functions such that $g \circ f$ is defined, then
 (a) $g \circ f$ is an increasing function
 (b) $g \circ f$ is a decreasing function
 (c) $g \circ f$ is neither increasing nor decreasing
 (d) none of these
37. In its domain, $f(x) = \frac{\sin^{-1} x}{\cos^{-1} x}$ is
 (a) an increasing function
 (b) a strictly increasing function
 (c) a decreasing function
 (d) a strictly decreasing function
38. Which of the following numbers is the maximum value of the function $f(x) = \frac{5 \sin^3 x \cos x}{\tan^2 x + 1}$, $x \in R$?
 (a) $5/8$ (b) $3/4$ (c) 1 (d) $5/2$
39. If $P(x_1, y_1), Q(x_2, y_2)$ be any two points on the curve $y = 3x - 2 - x^2$ for $1 < x < 2$, then maximum value of $3x_1 + 3x_2 - x_1^2 - x_2^2$ is
 (a) 9 (b) 4 (c) 2 (d) $9/2$
40. If θ is the angle (semi-vertical) of a cone of maximum volume and given slant height, then $\tan \theta$ is given by
 (a) 2 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$
41. Let a function $f(x)$ be defined as

$$f(x) = \begin{cases} \sin^{-1} \lambda + x^2; & 0 < x < 1 \\ 2x; & x \geq 1 \end{cases}$$

 $f(x)$ can have local minimum at $x = 1$ if the value of λ lies in the interval
 (a) $[\sin 1, 1]$ (b) $(-\sin 1, 1)$
 (c) $(\sin 1, 1]$ (d) $[0, \sin 1]$
42. The value of x for which $f(x) = \left(\sin \frac{\{x\}}{[x]} + \cos \frac{\{x\}}{[x]} \right)$ is maximum ($\{x\}$ and $[x]$ denotes fractional part and greatest integer part of x respectively)
 (a) $1 + \frac{\pi}{4}$ (b) $2 + \frac{\pi}{4}$
 (c) $1 - \frac{\pi}{4}$ (d) none of these
43. Let $f(x) = \begin{cases} 2x^2 + 2/x^2; & 0 < |x| \leq 2 \\ 1; & x = 0 \end{cases}$. Then, $f(x)$ has
 (a) least value 4 but no greatest value
 (b) greatest value 4
 (c) neither greatest or least value
 (d) least value 1 but no greatest value
44. If the function $f(x) = \frac{t + 3x - x^2}{x - 4}$, where ' t ' is a parameter that has a minimum and maximum, then the range of value of ' t ' is
 (a) $(0, 4)$ (b) $(0, \infty)$
 (c) $(-\infty, 4)$ (d) none of these

45. If $f(x) = x^{2/3}$ then

- (a) $(0, 0)$ is a point of maximum
- (b) $(0, 0)$ is not a point of minimum
- (c) $(0, 0)$ is a critical point
- (d) There is no critical point

SOLUTIONS

1. (c) : This problem requires a geometrical argument:

By similar triangles, $\frac{f(x)}{6} = \frac{x-a}{0-a} = \frac{g(x)}{3}$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{6}{3} = 2$$

$$2. (c) : \lim_{x \rightarrow 0} \left\{ \frac{f(1+x)}{f(1)} \right\}^{1/x} = \lim_{x \rightarrow 0} \left\{ 1 + \frac{f(1+x) - f(1)}{f(1)} \right\}^{1/x}$$

$$= e^{\lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x f(1)}} = e^{\frac{1}{f(1)} \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x}} = e^{\frac{f'(1)}{f(1)}}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} = f'(1) \right]$$

$$= e^{6/3} = e^2$$

$$3. (b) : \lim_{x \rightarrow \infty} \left\{ \frac{1^2}{1-x^3} + \frac{3}{1+x^2} + \frac{5^2}{1-x^3} + \frac{7}{1+x^2} + \dots \right\}$$

$$= \lim_{x \rightarrow \infty} \left\{ \frac{1^2 + 5^2 + 9^2 + \dots}{1-x^3} + \frac{3+7+11+\dots}{1+x^2} \right\}$$

$$= \lim_{x \rightarrow \infty} \left\{ \frac{\sum_{k=1}^x (4k-3)^2}{1-x^3} + \frac{\sum_{k=1}^x (4k-1)}{1+x^2} \right\}$$

$$= \lim_{x \rightarrow \infty} \left\{ \frac{16 \sum_{k=1}^x k^2 - 24 \sum_{k=1}^x k + \sum_{k=1}^x 9}{1-x^3} + \frac{4 \sum_{k=1}^x k - \sum_{k=1}^x 1}{1+x^2} \right\}$$

$$= \lim_{x \rightarrow \infty} \left\{ \frac{\frac{16x(x+1)(2x+1)}{6} - 12x(x+1) + 9x}{1-x^3} + \frac{2x(x+1) - x}{1+x^2} \right\}$$

$$= -\frac{32}{6} + 2 = -\frac{10}{3}$$

$$4. (b) : \lim_{x \rightarrow 0} \frac{(2^{\sin x} - 1)[\ln(1 + \sin 2x)]}{x^2 \frac{\tan^{-1} x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{2^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x} \times \frac{\ln(1 + \sin 2x)}{\sin 2x} \times \frac{\sin 2x}{2x} \times 2$$

$$= 2 \ln 2$$

5. (d) : As $x \rightarrow 0^-$, $[x] = -1$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1 + \sin(-1)}{-1} = -1 + \sin 1$$

whereas, if $x \rightarrow 0^+$ we get $[x] = 0$,

$$\therefore f(x) = 0 \Rightarrow \lim_{x \rightarrow 0^+} f(x) = 0$$

Thus, $\lim_{x \rightarrow 0} f(x)$ does not exist.

$$6. (d) : \lim_{n \rightarrow \infty} \frac{5^{n+1} + 3^n - 3^{2n}}{5^n + 2^n + 3^{2n+3}}$$

$$= \lim_{n \rightarrow \infty} \frac{5 \cdot 5^n + 3^n - 9^n}{5^n + 2^n + 27 \cdot 9^n} = \lim_{n \rightarrow \infty} \frac{5 \cdot \left(\frac{5}{9}\right)^n + \left(\frac{3}{9}\right)^n - 1}{\left(\frac{5}{9}\right)^n + \left(\frac{2}{9}\right)^n + 27} = \frac{-1}{27}$$

$$7. (b) : \text{Let } f'(0^+) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = k(\text{say})$$

$$\therefore f'(0^-) = \lim_{h \rightarrow 0} \frac{f(0) - f(0-h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(0) - f(h)}{h} = -k$$

$\therefore f'(0^+) \neq f'(0^-)$, but both are finite, so $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.

8. (d) : Note that $f(x) = 0$ for each integral value of x .

Also, if $0 \leq x < 1$, then $0 \leq x^2 < 1$

$\therefore [x] = 0$ and $[x^2] = 0 \Rightarrow f(x) = 0$ for $0 \leq x < 1$.

Next, if $1 \leq x < \sqrt{2}$, then

$1 \leq x^2 < 2 \Rightarrow [x] = 1$ and $[x^2] = 1$

Thus, $f(x) = [x]^2 - [x^2] = 0$ if $1 \leq x < \sqrt{2}$.

$\Rightarrow f(x) = 0$, if $0 \leq x < \sqrt{2}$.

This shows that $f(x)$ must be continuous at $x = 1$.

However, at points x other than integers and not lying between 0 and $\sqrt{2}$, $f(x) \neq 0$.

Thus, f is discontinuous at all integers except 1.

9. (b) : Clearly, $x = 1$ is a point of discontinuity of the

$$\text{function } f(x) = \frac{1}{1-x}$$

$$\text{If } x \neq 1, \text{ then } (f \circ f)(x) = f[f(x)] = f\left(\frac{1}{1-x}\right) = \frac{x-1}{x},$$

which is discontinuous at $x = 0$.

If $x \neq 0$ and $x \neq 1$, then

$$(f \circ f \circ f)(x) = f[(f \circ f)(x)] = f\left(\frac{x-1}{x}\right) = x$$

which is continuous everywhere.

So, the only points of discontinuity are $x = 0$ and $x = 1$.

10. (c) : Given, $f(x)$ is continuous on $\left(\frac{-\pi}{6}, \frac{\pi}{6}\right)$

$$\therefore b = f(0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\tan 2x / \tan 3x}$$

$$= \lim_{x \rightarrow 0^+} e^{\frac{\tan 2x}{2x} \times \frac{3x}{\tan 3x} \times \frac{2}{3}} = e^{2/3}$$

$$\text{Also, } b = f(0) = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{x \rightarrow 0^-} (1 + |\sin x|)^{\frac{1}{|\sin x|} \times a} = e^a$$

Therefore $a = 2/3$ and $b = e^{2/3}$.

$$\text{11. (c) : Since, } f(g(x)) = \begin{cases} 0, & \text{if } -\frac{\pi}{4} \leq x < 0 \\ 0, & \text{if } x \geq 0 \end{cases}$$

Which is always differentiable in $\left[-\frac{\pi}{4}, \infty\right)$ and also continuous.

12. (d) : At $x = \pm 1$, $x^2 - 1 = 0$ Hence, $f(x) = \frac{1}{x^2 - 1}$ function is discontinuous in R .

13. (d) : Basically $f(x) = 0 \quad \forall x \in R$.
So, it is continuous for all x .

$$\text{14. (a) : } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} |x-1| = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} e^x = 1 \text{ and } f(0) = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} |x-1| = 0$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} |x-1| = 0 \text{ and } f(1) = 0$$

$\therefore f(x)$ is continuous at $x = 0, 1$

$$\text{At } x = 0; \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{1-h-1}{h} = -1$$

$$\lim_{h \rightarrow 0^-} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0^-} \frac{e^{-h} - 1}{-h} = 1$$

$\therefore f(x)$ is not differentiable at $x = 0$

and $|x-1|$ is not differentiable at $x = 1$

$$\text{15. (b) : } f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{2x}{1+x^2}\right)^2}} \times \frac{(1+x^2) \cdot 2 - 2x(0+2x)}{(1+x^2)^2}$$

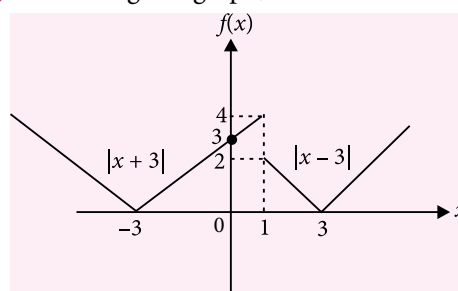
$$= \frac{1+x^2}{\sqrt{(1+x^2)^2 - 4x^2}} \times \frac{2+2x^2-4x^2}{(1+x^2)^2}$$

not defined when $(1+x^2)^2 - 4x^2 = 0$

$$\Rightarrow x = \pm 1$$

$\therefore f(x)$ is differentiable on $R - \{-1, 1\}$

16. (c) : Sketching the graph, we have



At three points, $x = -3, 1, 3$, $f(x)$ is not differentiable.

17. (c) : Obviously, AP is perpendicular on the tangent drawn to the curve.

18. (b) : For vertical tangents $\frac{dx}{d\theta} = 0$.

$$\text{So, we have } -3\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

Corresponding to these values of θ , we have

$$x = 2 - 3 \sin \frac{\pi}{2} = -1, y = 3 + 2 \cos \frac{\pi}{2} = 3;$$

$$x = 2 - 3 \sin \frac{3\pi}{2} = 5, y = 3 + 2 \cos \frac{3\pi}{2} = 3$$

Thus the required points are $(-1, 3), (5, 3)$.

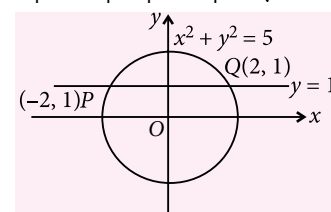
19. (a) : We know that $1 \leq |\sin x| + |\cos x| \leq \sqrt{2}$

So that $[|\sin x| + |\cos x|]$ will be constant function $y = 1$

Now intersection point P and Q are $(-2, 1)$ and $(2, 1)$ respectively.

Slope of line $y = 1$ is zero and slope of tangent at P and Q are (2) and (-2) respectively.

Thus the angle of intersection is $\tan^{-1}(2)$.



20. (b) : Using LMVT

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \leq \frac{1}{2} \Rightarrow f(x) \leq \frac{x}{2} \forall x \in (0, 2)$$

21. (a) : $f(x) = x^2 + x$ at $x = -2$

$$f'(x) = 2x + 1 \Rightarrow f'(-2) = -4 + 1 = -3$$

$$f(-2) = 4 - 2 = 2$$

\therefore Equation of tangent

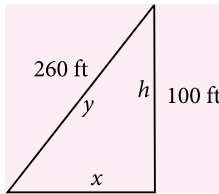
$$y - 2 = -3(x + 2) \Rightarrow 3x + y + 4 = 0$$

22. (b) : $x^2 + h^2 = y^2$, $\frac{dx}{dt} = \frac{13}{2}$ miles/hour

$$x \frac{dx}{dt} = y \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{\frac{13}{2} \sqrt{(260)^2 - (100)^2}}{260}$$

$$= \frac{13 \cdot 4 \cdot 60}{2 \cdot 260} = 6 \text{ miles/hour}$$



23. (a) : $f'(x) = (x - 2)(x - 4) + x(x - 4) + x(x - 2)$
 $= 3x^2 - 12x + 8$

Also $f(4) = 0$ and $f(1) = 3$

Thus $\frac{f(4) - f(1)}{4 - 1} = -1$. We must have $-1 = f'(x)$

$$\Rightarrow 3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow x = 1 \text{ or } x = 3.$$

24. (c) : Let $f(x) = ax^4 + bx^3 + cx^2 + dx$.

$$\begin{aligned} \text{Then, } f(0) &= 0 \text{ and } f(3) = 81a + 27b + 9c + 3d \\ &= 3(27a + 9b + 3c + d) \\ &= 0 (\because 27a + 9b + 3c + d = 0) \end{aligned}$$

Therefore 0 and 3 are roots of the polynomial $f(x)$. So by Rolle's theorem, there exists at least one root of the polynomial $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$ lying between 0 and 3.

25. (d) : Let α, β be two different roots of $f(x) = 0$ in $[0, 1]$ where $f(x) = x^3 - 3x + a$.

Therefore, by Rolle's theorem $f'(x) = 0$ has a root between α and β , i.e. in $(0, 1)$.

$$\text{But, } f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$\therefore f'(x) = 0 \text{ has no roots in } (0, 1)$$

Hence the given equation has no root lying between 0 and 1 for any value of a .

26. (b) : The equation of the curve is $y = e^{2x} + x^2$, when $x = 0 \Rightarrow y = 1$

$$\frac{dy}{dx} = 2e^{2x} + 2x = 2 \text{ at the point } (0, 1)$$

\therefore Slope of the normal $= -(1/2)$ and the normal passes through the point $(0, 1)$

$$\therefore \text{ The normal has the equation } y - 1 = -(1/2)x$$

$$\Rightarrow x + 2y - 2 = 0 \therefore \text{ Required distance} = 2/\sqrt{5}$$

$$\mathbf{27. (c) : } g'(x) = \frac{[1 - 4(f(x))^2]f'(x)}{\sqrt{1 - 2(f(x))^2}}$$

Now, as $f(x)$ and $g(x)$ are monotonically increasing.

$$f'(x) > 0 \text{ and } g'(x) > 0 \Rightarrow |f(x)| < 1/2$$

28. (b) : $f(x) = 2x^3 + ax^2 + bx - 3\cos^2 x$

$$f'(x) = 6x^2 + 2ax + b + 3\sin 2x > 0$$

$$\Rightarrow 6x^2 + 2ax + b - 3 > 0 \text{ as } \sin 2x \geq -1$$

$$\therefore 4a^2 - 24(b - 3) < 0 \Rightarrow a^2 - 6b + 18 < 0$$

29. (c) : $f(x) = \cos x - 2ax + b$

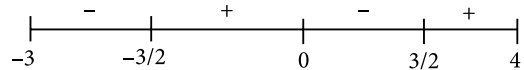
$$f'(x) = (-\sin x) - 2a \geq 0 \text{ or } \sin x + 2a \leq 0$$

$$\mathbf{30. (a) : } \phi'(x) = 3f'\left(\frac{x^2}{3}\right) \frac{2x}{3} - 2xf'(3 - x^2)$$

$$= 2x \left[f'\left(\frac{x^2}{3}\right) - f'(3 - x^2) \right]$$

$$\phi'(x) = 0 \Rightarrow x^2/3 = 3 - x^2 \Rightarrow x = \pm 3/2$$

$$f''(x) > 0 \Rightarrow f'(x) \text{ is increasing}$$



31. (a) : For $0 < x \leq \pi/2$, $[\cos x] = 0$. Hence $f(x) = 1$ for all $x \in (0, \pi/2]$. Trivially $f(x)$ is continuous on $(0, \pi/2)$. This function is neither strictly increasing nor strictly decreasing and its global maximum is 1.

32. (b) : $f(x) = 2x^3 - 3x^2 - 12x + 1$

$$f'(x) = 6(x^2 - x - 2) = 0 \text{ for } x = -1, x = 2$$

But $x = 2$ is outside the interval $[-1, 3/2]$

for $-1 < x < 3/2$; $f'(x) < 0$.

$\therefore f(x)$ is decreasing for $-1 \leq x \leq 3/2$

$$M = f(-1) = 8; m = f(3/2) = -17.$$

33. (c) : $f'(x) = 9x^8 + 21x^6 \geq 0 \forall x \in \mathbb{R}$

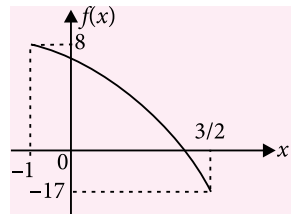
$f(x)$ is monotonic increasing for all real values of x

34. (c) : $|f(x) - f(y)| \leq (x - y)^2$

$$\lim_{y \rightarrow x} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{y \rightarrow x} |x - y|$$

$$\Rightarrow |f'(x)| \leq 0 \Rightarrow f'(x) = 0$$

$\therefore f(x)$ is a constant function



MPP-7 CLASS XI

ANSWER KEY

- | | | | | |
|-------------|------------|----------|----------|---------|
| 1. (d) | 2. (b) | 3. (b) | 4. (a) | 5. (c) |
| 6. (b) | 7. (a,b,d) | 8. (b,c) | 9. (b,c) | 10. (b) |
| 11. (a,b,c) | 12. (a,c) | 13. (c) | 14. (b) | 15. (c) |
| 16. (a) | 17. (2) | 18. (7) | 19. (0) | 20. (1) |

35. (c) : Function is increasing $\Rightarrow f'(x) \geq 0$

$$\Rightarrow 2\sec^2 x + (2a+1)\tan x + (a-2) \geq 0$$

$$\Rightarrow 2\tan^2 x + (2a+1)\tan x + a \geq 0$$

$$\Rightarrow (2a+1)^2 - 8a \leq 0$$

$$\Rightarrow (2a-1)^2 \leq 0 \Rightarrow 2a-1=0 \Rightarrow a=1/2$$

36. (a) : Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad [\because f \text{ is an increasing function}]$$

$$\Rightarrow g(f(x_1)) < g(f(x_2)) \quad [\because g \text{ is an increasing function}]$$

$$\Rightarrow g \circ f(x_1) < g \circ f(x_2)$$

Hence, $g \circ f$ is an increasing function.

$$37. (b) : f'(x) = \frac{\frac{1}{\sqrt{1-x^2}} \times (\cos^{-1} x) - \sin^{-1} x \times \left(-\frac{1}{\sqrt{1-x^2}}\right)}{(\cos^{-1} x)^2}$$

$$= \frac{\cos^{-1} x + \sin^{-1} x}{\sqrt{1-x^2} (\cos^{-1} x)^2} = \frac{\pi/2}{\sqrt{1-x^2} (\cos^{-1} x)^2} = +ve$$

$$\Rightarrow f'(x) > 0 \Rightarrow f(x) \text{ is an strictly increasing function}$$

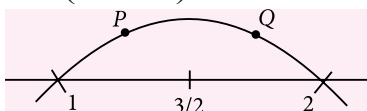
$$38. (a) : \text{Given, } f(x) = \frac{5\sin^3 x \cos x}{\tan^2 x + 1} = \frac{5\sin^3 x \cos x}{\frac{\sin^2 x}{\cos^2 x} + 1}$$

$$= 5(\sin^3 x)(\cos^3 x) = \frac{5}{8}\sin^3 2x$$

Hence, maximum value is $5/8$.

$$39. (d) : \frac{y_1 + y_2}{2} \leq \max. (y)$$

$$\Rightarrow y_1 + y_2 \leq 2 \left(\frac{9}{2} - 2 - \frac{9}{4} \right) \Rightarrow 3x_1 + 3x_2 - x_1^2 - x_2^2 \leq \frac{9}{2}$$



40. (c) : Let $OB = l$, $OA = l \cos \theta$ and $AB = l \sin \theta$ ($0 \leq \theta \leq \pi/2$). Then

$$V = \frac{\pi}{3} (AB)^2 (OA) = \frac{\pi}{3} l^3 \sin^2 \theta \cos \theta$$

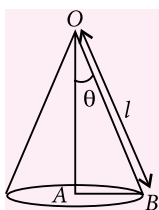
$$\Rightarrow \frac{dV}{d\theta} = \frac{\pi}{3} l^3 \sin \theta (3 \cos^2 \theta - 1)$$

So from $\frac{dV}{d\theta} = 0$, we get

$$\theta = 0 \text{ or } \cos \theta = \frac{1}{\sqrt{3}}.$$

$$\text{Also } V(0) = 0, V(\pi/2) = 0$$

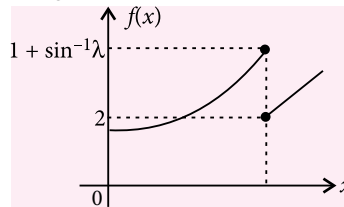
$$\text{and } V\left(\cos^{-1} \frac{1}{\sqrt{3}}\right) = \frac{2\pi l^3}{9\sqrt{3}}$$



Hence V is maximum when $\cos \theta = \frac{1}{\sqrt{3}}$.

$$\text{i.e., } \tan \theta = \sqrt{2}$$

41. (c) : For local minimum at $x = 1$, the graph should be same as in figure.



Hence $1 + \sin^{-1} \lambda > 2$

$$\Rightarrow \sin^{-1} \lambda > 1$$

$$\Rightarrow \lambda \in (\sin 1, 1]$$

$$42. (a) : \text{Let } \frac{\{x\}}{[x]} = \alpha$$

$$\Rightarrow f(x) = \sin \alpha + \cos \alpha = \sqrt{2} \left(\sin \left(\frac{\pi}{4} + \alpha \right) \right)$$

$f(x)$ is maximum at $\alpha = \pi/4$

$$\therefore \frac{\{x\}}{[x]} = \frac{\pi}{4} \Rightarrow \{x\} = \frac{\pi}{4} [x]$$

$$\text{It is true at } [x] = 1 \therefore \{x\} = \frac{\pi}{4}$$

$$\text{So } x = [x] + \{x\} = 1 + \frac{\pi}{4}$$

43. (d) : For $x \rightarrow 0$

$$2x^2 + \frac{2}{x^2} \rightarrow \infty. \text{ Also } 2 \left(x^2 + \frac{1}{x^2} \right) \geq 4$$

$$44. (c) : f(x) = \frac{t+3x-x^2}{x-4};$$

$$f'(x) = \frac{(x-4)(3-2x) - (t+3x-x^2)}{(x-4)^2}$$

For max. or min. $f'(x) = 0$

$$\Rightarrow -2x^2 + 11x - 12 - t - 3x + x^2 = 0$$

$$\Rightarrow -x^2 + 8x - (12+t) = 0$$

For one maxima and minima, $D > 0$

$$\Rightarrow 64 - 4(12+t) > 0$$

$$\Rightarrow 16 - 12 - t > 0 \Rightarrow 4 > t \text{ or } t < 4$$

45. (c) : $\frac{dy}{dx} = \frac{2}{3} x^{-1/3}$. This derivative is never zero, but

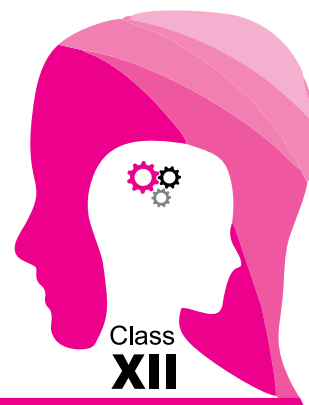
there is no derivative for $x = 0$.

So $(0, 0)$ is a critical point. If $x < 0$ then $\frac{dy}{dx} < 0$ and if

$x > 0$ then $\frac{dy}{dx} > 0$. Thus $(0, 0)$ is a point of minimum. ♦♦

CONCEPT BOOSTERS

Differentiation and Application of Derivatives



Class
XII

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

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DIFFERENTIATION

Let a be an interior point of the domain of a function $f(x)$. The derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \dots(i) \quad \text{provided the limit exists.}$$

If $f'(a)$ exists we say that $f(x)$ is differentiable at $x = a$, otherwise it is not differentiable.

Differentiation of Algebraic Functions

- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1} f'(x)$
- $\frac{d}{dx} \left(\frac{1}{x^n} \right) = -\frac{n}{x^{n+1}}$

Differentiation of Trigonometric Functions

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

Differentiation of Logarithmic and Exponential Functions

- $\frac{d}{dx} \log x = \frac{1}{x}, \text{ for } x > 0$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} a^x = a^x \log_e a, \text{ for } a > 0$

- $\frac{d}{dx} \log_a x = \frac{1}{x \log a}, \text{ for } x > 0, a > 0, a \neq 1$

Differentiation of Inverse Trigonometric Functions

- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$
- $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$
- $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{x^2-1}}, \text{ for } |x| > 1$
- $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{|x| \sqrt{x^2-1}}, \text{ for } |x| > 1$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, \text{ for } x \in \mathbb{R}$
- $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}, \text{ for } x \in \mathbb{R}$

THEOREMS FOR DIFFERENTIATION

Let $f(x)$, $g(x)$ and $u(x)$ be differentiable functions

- If at all points of a certain interval, $f'(x) = 0$, then the function $f(x)$ has a constant value within this interval.
- Chain Rule**
 - Case I : If y is a function of u and u is a function of x , then derivative of y with respect to x is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ or $y = f(u) \Rightarrow \frac{dy}{dx} = f'(u) \frac{du}{dx}$.
 - Case II : If y and x both are expressed in terms of t , y and x both are differentiable with respect to t , then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

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- **Sum and Difference Rule :** Using linear property

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

- **Product Rule**

$$(i) \quad \frac{d}{dx}(f(x) \cdot g(x)) = f(x) \frac{d}{dx}g(x) + g(x) \frac{d}{dx}f(x)$$

$$(ii) \quad \frac{d}{dx}(u \cdot v \cdot w) = u \cdot v \cdot \frac{dw}{dx} + v \cdot w \cdot \frac{du}{dx} + u \cdot w \cdot \frac{dv}{dx}$$

- **Scalar Multiple Rule :** $\frac{d}{dx}(k f(x)) = k \frac{d}{dx}f(x)$
- **Quotient Rule :**

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx}(f(x)) - f(x) \frac{d}{dx}(g(x))}{(g(x))^2}$$

provided $g(x) \neq 0$.

METHODS OF DIFFERENTIATION

- **Differentiation of Implicit Functions :** If y is expressed entirely in terms of x , then we say that y is an explicit function of x . For example $y = \sin x$, $y = e^x$, $y = x^2 + x + 1$ etc. If y is related to x but cannot be conveniently expressed in the form of $y = f(x)$ but can be expressed in the form $f(x, y) = 0$, then we say that y is an implicit function of x .

- Differentiate each term of $f(x, y) = 0$ with respect to x .
- Collect the terms containing dy/dx on one side and the terms not involving dy/dx on the other side.
- Express dy/dx as a function of x or y or both.

- **Logarithmic Differentiation :** If differentiation of an expression or an equation is done after taking log on both sides, then it is called logarithmic differentiation. This method is useful for the function having following forms

$$(i) \quad y = [f(x)]^{g(x)}$$

$$(ii) \quad y = \frac{f_1(x) \cdot f_2(x) \cdot \dots}{g_1(x) \cdot g_2(x) \cdot \dots} \text{ where } g_i(x) \neq 0$$

(where $i = 1, 2, 3, \dots$), $f_i(x)$ and $g_i(x)$ both are differentiable.

(i) Case I : $y = [f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x . To find the derivative of this type of functions we proceed as follows: Let $y = [f(x)]^{g(x)}$. Taking logarithm on both the sides, we have $\log y = g(x) \cdot \log f(x)$ and then we differentiate w.r.t. x .

$$(ii) \text{ Case II : } y = \frac{f_1(x) \cdot f_2(x)}{g_1(x) \cdot g_2(x)}$$

Taking logarithm on both the sides, we have $\log y = \log[f_1(x)] + \log[f_2(x)] - \log[g_1(x)] - \log[g_2(x)]$ and then we differentiate w.r.t. x .

- **Differentiation of Infinite Series :** If y is given in the form of infinite series of x and we have to find out $\frac{dy}{dx}$, then we remove one or more terms, it does not affect the series.

$$(i) \quad \text{If } y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots}}},$$

$$\text{then } y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$$

$$(ii) \quad \text{If } y = f(x)^{f(x)^{f(x)^{\dots}}}, \text{ then } y = f(x)^y$$

$$(iii) \quad \text{If } y = f(x) + \frac{1}{f(x) + \frac{1}{f(x) + \dots}}, \text{ then } y = f(x) + \frac{1}{y}.$$

- **Differentiation of Composite Function :** Suppose a function is given in form of $f \circ g(x)$ or $f[g(x)]$. Differentiate applying chain rule,

$$\frac{d}{dx} f[g(x)] = f'[g(x)] \cdot g'(x)$$

SUCCESSIVE DIFFERENTIATION OR HIGHER ORDER DERIVATIVES

- If y is a function of x and is differentiable with respect to x , then its derivative $\frac{dy}{dx}$ can be found which is known as derivative of first order. If the first derivative $\frac{dy}{dx}$ is also a differentiable function, then it can be further differentiated with respect to x and this derivative is denoted by d^2y/dx^2 , which is called the second derivative of y with respect to x . Similarly n^{th} derivative of y is denoted by $\frac{d^n y}{dx^n}$. All these derivatives are called as successive derivatives and this process is known as successive differentiation. If $y = f(x)$, then the value of the n^{th} order derivative at $x = a$ is usually denoted

$$\text{by } \left(\frac{d^n y}{dx^n} \right)_{x=a} \text{ or } (y_n)_{x=a}$$

n^{th} Derivatives of Some Standard Functions

- $\frac{d^n}{dx^n} \sin(ax + b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$
- $\frac{d^n}{dx^n} \cos(ax + b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$
- $\frac{d^n}{dx^n} (ax + b)^m = \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}$, where $m > n$, $m \in \mathbb{N}$

- $\frac{d^n}{dx^n} \log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
- $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$ • $\frac{d^n(a^x)}{dx^n} = a^x (\log a)^n$
- $\frac{d^n}{dx^n} e^{ax} \sin(bx+c) = r^n e^{ax} \sin(bx+c+n\phi)$
- $\frac{d^n}{dx^n} e^{ax} \cos(bx+c) = r^n e^{ax} \cos(bx+c+n\phi)$
where $r = \sqrt{a^2 + b^2}$; $\phi = \tan^{-1} \frac{b}{a}$

DIFFERENTIATION OF INTEGRAL FUNCTION

- If $g_1(x)$ and $g_2(x)$ both functions are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous for $g_1(a) \leq f(t) \leq g_2(b)$, then

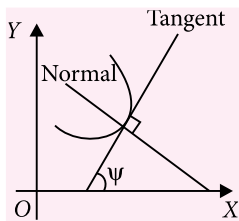
$$\frac{d}{dx} \int_{g_1(x)}^{g_2(x)} f(t) dt = f[g_2(x)] \frac{d}{dx} g_2(x) - f[g_1(x)] \frac{d}{dx} g_1(x)$$

SLOPE OF THE TANGENT AND NORMAL

- **Slope of the Tangent :** If a tangent is drawn to the curve $y = f(x)$ at a point $P(x_1, y_1)$ and this tangent makes an angle ψ with positive x -direction, then

$$\left(\frac{dy}{dx} \right)_{(x_1, y_1)} = \tan \psi = \text{Slope of the tangent.}$$
- **Slope of the Normal :** The normal to a curve at a point $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P . Slope of the normal

$$= \frac{-1}{\text{Slope of tangent}} = - \left(\frac{dx}{dy} \right)_{(x_1, y_1)}$$



EQUATION OF THE TANGENT AND NORMAL

- **Equation of the Tangent :** The equation of the tangent to the curve $y = f(x)$ at a point $P(x_1, y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} (x - x_1)$$
- **Equation of the Normal :** The equation of the normal to the curve $y = f(x)$ at a point $P(x_1, y_1)$ is

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx} \right)_{(x_1, y_1)}} (x - x_1)$$

ANGLE OF INTERSECTION OF TWO CURVES

- The angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection. Thus, the angle between the tangents of the two curves $y = f_1(x)$ and $y = f_2(x)$ is given by

$$\tan \phi = \frac{\left| \left(\frac{dy}{dx} \right)_{1(x_1, y_1)} - \left(\frac{dy}{dx} \right)_{2(x_1, y_1)} \right|}{1 + \left(\frac{dy}{dx} \right)_{1(x_1, y_1)} \left(\frac{dy}{dx} \right)_{2(x_1, y_1)}}$$

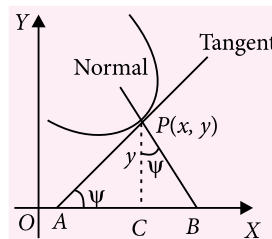
- **Orthogonal Curves :** If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally. The curves are called orthogonal curves. If the curves are orthogonal, then

$$\phi = \frac{\pi}{2}; m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx} \right)_1 \left(\frac{dy}{dx} \right)_2 = -1$$

LENGTH OF TANGENT, NORMAL, SUBTANGENT, SUBNORMAL

- Length of tangent, $PA = y \operatorname{cosec} \psi$

$$= y \frac{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}}{\left(\frac{dy}{dx} \right)}$$



- Length of normal, $PB = y \sec \psi = y \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$
- Length of sub-tangent, $AC = y \cot \psi = \frac{y}{\left(\frac{dy}{dx} \right)}$
- Length of subnormal, $BC = y \tan \psi = y \left(\frac{dy}{dx} \right)$

INCREASING AND DECREASING FUNCTIONS

- A function f is said to be an increasing function in $]a, b[$, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in]a, b[$.
- A function f is said to be a decreasing function in $]a, b[$, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$, $\forall x_1, x_2 \in]a, b[$.
- $f(x)$ is known as non-decreasing if $f'(x) \geq 0$ and non-increasing if $f'(x) \leq 0$.

MONOTONIC FUNCTION

A function f is said to be monotonic in an interval if it is either increasing or decreasing in that interval.

We summarize the results in the table below :

Properties of Monotonic Functions

- If $f(x)$ is a strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.
- If $f(x)$ is a strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.
- If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ [$f'(c) > 0$] for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on $[a, b]$.
- If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ [$f'(c) < 0$] for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) decreasing function on $[a, b]$.
- If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing function on $[a, b]$.
- If one of the two functions $f(x)$, $g(x)$ is strictly (or monotonically) increasing and other is strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing on $[a, b]$.

MAXIMUM AND MINIMUM

- A function $f(x)$ is said to attain a maximum at $x = a$ if there exists a neighbourhood $(a - \delta, a + \delta)$ such that $f(x) < f(a)$ for all $x \in (a - \delta, a + \delta)$, $x \neq a$
 $\Rightarrow f(x) - f(a) < 0$ for all $x \in (a - \delta, a + \delta)$, $x \neq a$
 In such a case, $f(a)$ is said to be the maximum value of $f(x)$ at $x = a$.
- A function $f(x)$ is said to attain a minimum at $x = a$ if there exists a $(a - \delta, a + \delta)$ such that $f(x) > f(a)$ for all $x \in (a - \delta, a + \delta)$, $x \neq a$
 $\Rightarrow f(x) - f(a) > 0$ for all $x \in (a - \delta, a + \delta)$, $x \neq a$
 In such a case, $f(a)$ is said to be the minimum value of $f(x)$ at $x = a$.
- The points at which a function attains either the maximum values or the minimum values are known as the extreme points or turning points and both maximum and minimum values of $f(x)$ are called extreme or extreme values.
 Thus a function attains an extreme value at $x = a$ if $f(a)$ is either a maximum or a minimum value. Consequently at an extreme point a , $f(x) - f(a)$ keeps the same sign for all values of x in a deleted nbd of a .

SUFFICIENT CRITERIA FOR EXTREME VALUES (1st DERIVATIVE TEST)

Let $f(x)$ be a function differentiable at $x = a$.

Then (a) $x = a$ is a point of local maximum of $f(x)$ if

- $f'(a) = 0$ and
- $f'(a)$ changes sign from positive to negative as x passes through a i.e., $f'(x) > 0$ at every point in the left neighbourhood $(a - \delta, a)$ of a and $f'(x) < 0$ at every point in the right neighbourhood $(a, a + \delta)$ of a .

(b) $x = a$ is a point of local minimum of $f(x)$ if

- $f'(a) = 0$ and
- $f'(a)$ changes sign from negative to positive as x passes through a , i.e., $f'(x) < 0$ at every point in the left neighbourhood $(a - \delta, a)$ of a and $f'(x) > 0$ at every point in the right neighbourhood $(a, a + \delta)$ of a .

(c) If $f'(a) = 0$ but $f'(a)$ does not change sign, that is, has the same sign in the complete neighbourhood of a , then a is neither a point of local maximum nor a point of local minimum.

Working Rule for Determining Extreme Values of a Function $f(x)$

Step I : Put $y = f(x)$

Step II : Find $\frac{dy}{dx}$

Step III : Put $\frac{dy}{dx} = 0$ and solve this equation for x . Let

$x = c_1, c_2, \dots, c_n$ be values of x obtained by putting

$\frac{dy}{dx} = 0$. c_1, c_2, \dots, c_n are the stationary values of x .

Step IV : Consider $x = c_1$. If $\frac{dy}{dx}$ changes its sign from positive to negative as x passes through c_1 , then the function attains a local maximum at $x = c_1$. If $\frac{dy}{dx}$ changes its sign from negative to positive as x passes through c_1 , then the function attains a local minimum at $x = c_1$. In case there is no change of sign, then $x = c_1$ is neither a point of local maximum nor a point of local minimum.

HIGHER ORDER DERIVATIVE TEST

- Find $f'(x)$ and equate it to zero. Solve $f'(x) = 0$ let its roots are $x = a_1, a_2, \dots$
- Find $f''(x)$ and at $x = a_1$,
 - If $f''(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.
 - If $f''(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$.
- If at $x = a_1$, $f''(a_1) = 0$, then find $f'''(x)$. If $f'''(a_1) \neq 0$, then $f(x)$ is neither maximum nor minimum at $x = a_1$.

If $f'''(a_1) = 0$, then find $f^{iv}(x)$.

If $f^{iv}(x)$ is +ve (Minimum value)

$f^{iv}(x)$ is - ve (Maximum value)

- If at $x = a_1$, $f^{iv}(a_1) = 0$, then find $f^v(x)$ and proceed similarly.

PROPERTIES OF MAXIMA AND MINIMA

(i) Maxima and minima occur alternately, that is between two maxima there is one minimum and vice-versa.

(ii) If $f(x) \rightarrow \infty$ as $x \rightarrow a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b , then $f(c)$ is necessarily the minimum and the least value.

If $f(x) \rightarrow -\infty$ as $x \rightarrow a$ or b , then $f(c)$ is necessarily the maximum and the greatest value.

ROLLE'S THEOREM

If a function $f(x)$ is such that,

(i) It is continuous in the closed interval $[a, b]$

(ii) It is derivable in the open interval (a, b)

(iii) $f(a) = f(b)$

Then there exists at least one value ' c ' of x in the open interval (a, b) such that $f'(c) = 0$.

LAGRANGE'S MEAN VALUE THEOREM

If a function $f(x)$ is such that,

(i) It is continuous in the closed interval $[a, b]$

(ii) It is derivable in the open interval (a, b)

Then there exists at least one value ' c ' of x in the open interval (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

PROBLEMS

Single Correct Answer Type

1. If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$, then $\frac{dy}{dx} =$

- (a) y (b) $y + \frac{x^n}{n!}$
(c) $y - \frac{x^n}{n!}$ (d) $y - 1 - \frac{x^n}{n!}$

2. $\frac{d}{dx} \left(\tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) \right) =$

- (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) -1 (d) 1

3. $\frac{d}{dx} \log |x| = \dots, (x \neq 0)$

- (a) $\frac{1}{x}$ (b) $-\frac{1}{x}$ (c) x (d) $-x$

4. If $y = a \sin x + b \cos x$, then $y^2 + \left(\frac{dy}{dx} \right)^2$ is a

- (a) function of x (b) function of y
(c) function of x and y (d) constant

5. $\frac{d}{dx} \left(\tan^{-1} \sqrt{\frac{1 + \cos \frac{x}{2}}{1 - \cos \frac{x}{2}}} \right)$ is equal to

- (a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $\frac{1}{4}$

6. If $y = \log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$, then $\frac{dy}{dx} =$

- (a) $\frac{1}{x \log_e 10} - \frac{\log_e 10}{x (\log_e x)^2}$
(b) $\frac{1}{x \log_e 10} - \frac{1}{x \log_{10} e}$
(c) $\frac{1}{x \log_e 10} + \frac{\log_e 10}{x (\log_e x)^2}$
(d) none of these

7. If $y = \log \left(\frac{1+x}{1-x} \right)^{1/4} - \frac{1}{2} \tan^{-1} x$, then $\frac{dy}{dx} =$

- (a) $\frac{x^2}{1-x^4}$ (b) $\frac{2x^2}{1-x^4}$
(c) $\frac{x^2}{2(1-x^4)}$ (d) none of these

8. If $y = \tan^{-1} \frac{4x}{1+5x^2} + \tan^{-1} \frac{2+3x}{3-2x}$, then $\frac{dy}{dx} =$

- (a) $\frac{1}{1+25x^2} + \frac{2}{1+x^2}$ (b) $\frac{5}{1+25x^2} + \frac{2}{1+x^2}$
(c) $\frac{5}{1+25x^2}$ (d) $\frac{1}{1+25x^2}$

9. If $f(x) = \sqrt{1 + \cos^2(x^2)}$, then $f' \left(\frac{\sqrt{\pi}}{2} \right)$ is

- (a) $\sqrt{\pi}/6$ (b) $-\sqrt{(\pi/6)}$
(c) $1/\sqrt{6}$ (d) $\pi/\sqrt{6}$

10. $\frac{d}{dx} [(1+x^2) \tan^{-1} x] =$

- (a) $x \tan^{-1} x$ (b) $2 \tan^{-1} x$
(c) $2x \tan^{-1} x + 1$ (d) $x \tan^{-1} x + 1$

11. $\frac{d}{dx} \log(\sqrt{x-a} + \sqrt{x-b}) =$

- (a) $\frac{1}{2[\sqrt{(x-a)} + \sqrt{(x-b)}]}$ (b) $\frac{1}{2\sqrt{(x-a)(x-b)}}$
(c) $\frac{1}{\sqrt{(x-a)(x-b)}}$ (d) none of these

12. $\frac{d}{dx} \tan^{-1} \frac{4\sqrt{x}}{1-4x} =$

- (a) $\frac{1}{\sqrt{x}(1+4x)}$ (b) $\frac{2}{\sqrt{x}(1+4x)}$
(c) $\frac{4}{\sqrt{x}(1+4x)}$ (d) none of these

13. If $y = \sec^{-1} \left(\frac{\sqrt{x}+1}{\sqrt{x}-1} \right) + \sin^{-1} \left(\frac{\sqrt{x}-1}{\sqrt{x}+1} \right)$, then $\frac{dy}{dx} =$

- (a) 0 (b) $\frac{1}{\sqrt{x}+1}$
(c) 1 (d) none of these

14. If $y = \sqrt{\frac{1+\tan x}{1-\tan x}}$, then $\frac{dy}{dx} =$

- (a) $\frac{1}{2} \sqrt{\frac{1-\tan x}{1+\tan x}} \cdot \sec^2 \left(\frac{\pi}{4} + x \right)$
(b) $\sqrt{\frac{1-\tan x}{1+\tan x}} \cdot \sec^2 \left(\frac{\pi}{4} + x \right)$
(c) $\frac{1}{2} \sqrt{\frac{1-\tan x}{1+\tan x}} \cdot \sec \left(\frac{\pi}{4} + x \right)$
(d) none of these

15. If $y = \frac{e^x \log x}{x^2}$, then $\frac{dy}{dx} =$

- (a) $\frac{e^x [1+(x+2) \log x]}{x^3}$ (b) $\frac{e^x [1-(x-2) \log x]}{x^4}$
(c) $\frac{e^x [1-(x-2) \log x]}{x^3}$ (d) $\frac{e^x [1+(x-2) \log x]}{x^3}$

16. If $r = [2\phi + \cos^2(2\phi + \pi/4)]^{1/2}$ then what is the value of the derivative $dr/d\phi$ at $\phi = \pi/4$

- (a) $2 \left(\frac{1}{\pi+1} \right)^{1/2}$ (b) $2 \left(\frac{2}{\pi+1} \right)^{-1/2}$
(c) $2 \left(\frac{1}{\pi+1} \right)^{-1/2}$ (d) $2 \left(\frac{2}{\pi+1} \right)^{1/2}$

17. If $y = \tan^{-1} \left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right)$ then $\frac{dy}{dx} =$

- (a) 2 (b) -1 (c) $\frac{a}{b}$ (d) 0

18. $\frac{d}{dx} (\log \tan x) =$

- (a) $2 \sec 2x$ (b) $2 \operatorname{cosec} 2x$
(c) $\sec 2x$ (d) $\operatorname{cosec} 2x$

19. $\frac{d}{dx} \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 =$

(a) $1 - \frac{1}{x^2}$

(b) $1 + \frac{1}{x^2}$

(c) $1 - \frac{1}{2x}$

(d) none of these

20. If $y = x + \frac{1}{x}$, then

(a) $x^2 \frac{dy}{dx} + xy = 0$

(b) $x^2 \frac{dy}{dx} + xy + 2 = 0$

(c) $x^2 \frac{dy}{dx} - xy + 2 = 0$

(d) none of these

21. $\frac{d}{dx} (e^x \log \sin 2x) =$

(a) $e^x (\log \sin 2x + 2 \cot 2x)$

(b) $e^x (\log \cos 2x + 2 \cot 2x)$

(c) $e^x (\log \cos 2x + \cot 2x)$

(d) none of these

22. If $y = \tan^{-1} \frac{x}{1+\sqrt{1-x^2}} + \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\}$,

then $\frac{dy}{dx} =$

(a) $\frac{x}{\sqrt{1-x^2}}$

(b) $\frac{1-2x}{\sqrt{1-x^2}}$

(c) $\frac{1-2x}{2\sqrt{1-x^2}}$

(d) $\frac{1}{1+x^2}$

23. $\frac{d}{dx} \left[\sin^2 \cot^{-1} \left\{ \sqrt{\frac{1-x}{1+x}} \right\} \right]$ equals

(a) -1

(b) $\frac{1}{2}$

(c) $-\frac{1}{2}$

(d) 1

24. If $y = \tan^{-1} \left(\frac{x}{1+\sqrt{1-x^2}} \right)$, then $\frac{dy}{dx} =$

(a) $\frac{1}{2\sqrt{1-x^2}}$

(b) $1 - \sqrt{1-x^2}$

(c) $\frac{1}{2}$

(d) $\frac{1}{\sqrt{1-x^2}}$

25. Let $3f(x) - 2f(1/x) = x$, then $f'(2)$ is equal to

(a) $2/7$

(b) $1/2$

(c) 2

(d) $7/2$

26. The differential coefficient of

$\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$ is

(a) $\sqrt{1-x^2}$

(b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{2\sqrt{1-x^2}}$

(d) x

27. If $y = A \cos nx + B \sin nx$, then $\frac{d^2 y}{dx^2} =$
 (a) $n^2 y$ (b) $-y$
 (c) $-n^2 y$ (d) none of these
28. If $f(x) = a \sin(\log x)$, then $x^2 f''(x) + x f'(x) =$
 (a) $f(x)$ (b) $-f(x)$ (c) 0 (d) 1
29. If for a function $f(x)$, $f'(a) = 0, f''(a) = 0, f'''(a) > 0$, then at $x = a, f(x)$ is
 (a) minimum (b) maximum
 (c) not an extreme point (d) extreme point
30. If $f(x) = 2x^3 - 21x^2 + 36x - 30$, then which one of the following is correct?
 (a) $f(x)$ has minimum at $x = 1$
 (b) $f(x)$ has maximum at $x = 6$
 (c) $f(x)$ has maximum at $x = 1$
 (d) $f(x)$ has no maxima or minima
31. The function $x + \frac{1}{x}, (x \neq 0)$ is a non-increasing function in the interval
 (a) $[-1, 1]$ (b) $[0, 1]$ (c) $[-1, 0]$ (d) $[-1, 2]$
32. The function $\sin x - bx + c$ will be increasing in the interval $(-\infty, \infty)$, if
 (a) $b \leq 1$ (b) $b \leq 0$ (c) $b < -1$ (d) $b \geq 0$
33. The function $\sin x - \cos x$ is increasing in the interval
 (a) $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$ (b) $\left[0, \frac{3\pi}{4}\right]$
 (c) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (d) none of these
34. The function $f(x) = \frac{1}{5^x}$ is
 (a) decreasing for all x (b) increasing for all x
 (c) neither increasing nor decreasing
 (d) increasing for $x > 0$ and decreasing for $x < 0$
35. The value of 'a' in order that $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ decreases for all real values of x , is given by
 (a) $a < 1$ (b) $a \geq 1$
 (c) $a \geq \sqrt{2}$ (d) $a < \sqrt{2}$

Multiple Correct Answer Type

36. If $f(x) = \left(\frac{a^2 - 1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$ is monotonic increasing for every $x \in R$ then 'a' lies in
 (a) $(1, 2)$ (b) $(1, \infty)$
 (c) $(-\infty, -3)$ (d) $(-10, -7)$

37. Consider the following statements

Statement - I: If $0 < \alpha < \beta < \frac{\pi}{2}$, then $\frac{\tan \beta}{\tan \alpha} > \frac{\alpha}{\beta}$

Statement - II: If $x \geq 0$, then $\frac{x}{1+x} \leq \log(1+x) \leq x$.

Then

- (a) I is true (b) I is false
 (c) II is true (d) II is false
38. Let $f(x) = \frac{e^x}{1+x^2}$ and $g(x) = f'(x)$ then
 (a) $g(x)$ has four points of local extremum
 (b) $g(x)$ has two points of local extremum
 (c) $g(x)$ has a point of local minimum at $x = 1$
 (d) $g(x)$ has a point of local maximum at some $x \in (-1, 0)$

39. Let $f(x) = x^2 \cdot e^{-x^2}$ then
 (a) $f(x)$ has local maxima at $x = -1$ and $x = 1$
 (b) $f(x)$ has local minima at $x = 0$
 (c) $f(x)$ is strictly decreasing on $x \in R$
 (d) Range of $f(x)$ is $[0, 1/e]$

40. Let $f(x) = \begin{cases} 2x - 4, & x \leq 2 \\ -x^2 + \frac{k^3(k-1)^2}{k^2 - k - 2} + 4, & x > 2 \end{cases}$,
 $f(x)$ attains local maximum at $x = 2$ if k lies in
 (a) $(0, 1)$ (b) $(3, \infty)$ (c) $(-\infty, -1)$ (d) $(1, 2)$

41. If $f(x) = \begin{cases} |x+1| & ; -2 < x < 0 \\ \sqrt[3]{1-x} & ; 0 < x < 1 \\ 2 & ; x = 0 \\ \sqrt{x+1} & ; x \geq 1 \end{cases}$. Then $f(x)$
 (a) has neither maximum nor minimum at $x = 0$
 (b) has maximum at $x = 0$
 (c) has neither maximum nor minimum at $x = 1$
 (d) no global maximum

42. Which of the following are true for $\forall x \in (0, \infty)$?
 (a) $\ln(1+x) > x - \frac{x^2}{2}$ (b) $\ln(1+x) > \frac{x}{x+1}$
 (c) $4 \cos x + x > 0$ (d) $2 \tan^{-1} x < x + 1$

43. If $\log_2 \left(\log_{\frac{1}{2}} (\log_2(x)) \right) = \log_3 \left(\log_{\frac{1}{3}} (\log_3(y)) \right)$
 $= \log_5 \left(\log_{\frac{1}{5}} (\log_5(z)) \right) = 0$ for positive x, y and z , then which of the following is/are not true?

CONCEPT MAP

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

Class XI

Quadratic Equation

An equation of the form $ax^2 + bx + c$ is called a quadratic equation where $a, b, c \in \mathbb{R}$ if $b^2 - 4ac < 0$, then the solution is given as

$$x = \frac{-b \pm \sqrt{4ac - b^2}i}{2a}$$

De-Moivre's Theorem

$$z = r(\text{cis } \theta), z^n = r^n(\text{cis } n\theta)$$

Also, n^{th} roots of unity is given by $z^n = 1$ and

$$z = \text{cis} \left(\frac{2k\pi}{n} \right),$$

$$k = 0, 1, 2, \dots, n-1$$

if $n = 3$, $z = 1, \omega, \omega^2$ are cube roots of unity.

Euler's Form

$$z = re^{i\theta}, \bar{z} = re^{-i\theta}$$

where $-\pi < \theta \leq \pi$, θ is the principal argument.

Forms

Complex Number

A number of the form $z = a + ib$ (where a = Real part, b = Imaginary part and $a, b \in \mathbb{R}$) is defined to be a complex number.

Polar Form

$$z = a + ib$$

$$= r(\cos\theta + i\sin\theta) = r\text{cis } \theta$$

where r = modulus of z

$$= \sqrt{a^2 + b^2},$$

and θ = angle with x -axis in +ve direction

$$\text{i.e., } \theta = \tan^{-1} \left(\frac{b}{a} \right)$$

Algebra of Complex Numbers

$$\text{Let } z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$$

$$\text{Addition : } z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$\text{Subtraction : } z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

$$\text{Multiplication : } z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\text{Division : } \frac{z_1}{z_2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2},$$

where $z_2 \neq 0$

$$\text{Equality : } z_1 = z_2 \Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$

$$\text{Multiplicative Inverse : } z^{-1} = \frac{1}{z} = \frac{a_1}{a_1^2 + b_1^2} + i \frac{(-b_1)}{a_1^2 + b_1^2}$$

Conjugate

$$\text{Conjugate of } z = a + ib \text{ is } \bar{z} = a - ib$$

Properties

- $(\bar{\bar{z}}) = z$
- $\text{Re}(z) = \frac{z + \bar{z}}{2}; \text{Im } z = \frac{z - \bar{z}}{2}$
- $z = \bar{z} \Leftrightarrow z$ is purely real
- $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2; \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2; \overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2 (\bar{z}_2 \neq 0)$
- $\overline{(z^n)} = (\bar{z})^n$
- $\alpha = f(z) \Rightarrow \bar{\alpha} = f(\bar{z}), \alpha \in \mathbb{C}$

Some Terms of Complex Numbers

Argument

Argument of z is the angle between +ve real axis to the line joining the point to the origin.

Properties

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2)$
- $\arg(z^n) = n \arg(z)$

Modulus

$$\text{Modulus of } z = a + ib \text{ is } |z| = \sqrt{a^2 + b^2}$$

$$|z| = 0 \Leftrightarrow z = 0$$

Properties

- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1| |z_2|; |z_1 / z_2| = |z_1| / |z_2| (z_2 \neq 0)$
- $|z^n| = |z|^n$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$
- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1 \bar{z}_2)$

MATRICES

CONCEPT MAP

Class XII

Matrices

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or entries of the matrix.

Order of a Matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$.

	Operations	Properties
Addition and Subtraction	$A \pm B = C$ <i>i.e.</i> , $[a_{ij}]_{m \times n} \pm [b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$	<ul style="list-style-type: none"> $A + B = B + A$ $A + (B + C) = (A + B) + C$ Additive inverse $= -A$ Additive identity $= O$
Multiplication	$A_{m \times k} \times B_{k \times q} = C_{m \times q}$ <i>i.e.</i> , $\left[\sum_{r=1}^k a_{ir} b_{rj} \right] = [c_{ij}]$	<ul style="list-style-type: none"> AB exist $\nRightarrow BA$ exists AB may or may not be equal to BA $(AB)C = A(BC)$ $I_m \times A_{m \times n} = A_{m \times n} = A_{m \times n} \times I_n$ $A(B + C) = AB + AC$; $(B + C)A = BA + CA$
Scalar Multiplication	$kA = B$ <i>i.e.</i> , $[k a_{ij}] = [b_{ij}]$	<ul style="list-style-type: none"> $k(A + B) = kA + kB$ $(k + m)A = kA + mA$

Transpose

Transpose is obtained by interchanging rows and columns. If $A = [a_{ij}]_{m \times n}$, then A' or $A^T = [a_{ji}]_{n \times m}$

$$(A')' = A$$

$$(AB)' = B'A'$$

Properties

$$(A \pm B)' = A' \pm B'$$

$$|A'| = |A|$$

Types of Matrix

Column Matrix : $A = [a_{ij}]_{m \times 1}$

Row Matrix : $A = [a_{ij}]_{1 \times n}$

Square Matrix : $A = [a_{ij}]_{m \times m}$

Diagonal Matrix : $A = [a_{ij}]_{m \times m}$
where $a_{ij} = 0 \forall i \neq j$

Scalar Matrix : $A = [a_{ij}]_{n \times n}$
where $a_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ k, & \text{if } i = j \end{cases}$
for some constant k

Zero Matrix : $A = [a_{ij}]$,
where $a_{ij} = \{0 \forall i = j \text{ and } i \neq j\}$

Identity Matrix : $A = [a_{ij}]$
where $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Special Matrices

Nilpotent Matrix : $A^k = 0$ and $A^{k-1} \neq 0$, $k \in \mathbb{Z}^+$
 $\Rightarrow |A| = 0$, order = Least value of k

Involutory Matrix : $A^2 = I \Rightarrow |A| = \pm 1$

Orthogonal Matrix : $AA^T = A^T A = I$
 $\Rightarrow |A| = \pm 1$

Periodic Matrix : $A^k = A$
 $\Rightarrow |A| = 0, 1$, order $= k - 1$

Idempotent Matrix : $A^2 = A \Rightarrow |A| = 0, 1$

Unitary Matrix : $AA^0 = A^0 A = I$

Symmetric Matrix : $A' = A$

Skew Symmetric Matrix : $A' = -A$
[where A is square matrix]

Inverse

If A and B are two square matrices such that $AB = BA = I$, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B .

Properties

A^{-1} exists iff $|A| \neq 0$

$(A^{-1})^{-1} = A$

$(kA)^{-1} = A^{-1}/k$

$(AB)^{-1} = B^{-1}A^{-1}$

$|A^{-1}| = (|A|)^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

- (a) $z < x < y$ (b) $x < y < z$
 (c) $y < z < x$ (d) $z < y < x$

44. For the function $f(x) = x \cos \frac{1}{x}$, $x \geq 1$
 (a) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$
 (b) $\lim_{x \rightarrow \infty} f'(x) = 1$
 (c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$
 (d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$
45. If $f''(x) > 0 \forall x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2 \tan x + 4)$, $0 < x < \pi/2$, then, $g(x)$ is increasing in
 (a) $(0, \pi/4)$ (b) $(\pi/6, \pi/3)$
 (c) $(0, \pi/3)$ (d) $(\pi/4, \pi/2)$

Comprehension Type

Paragraph for Q.No. 46 to 48

For a polynomial function $y = f(x)$

Points of extrema are obtained at points where $f'(x) = 0$

$f''(x_1) > 0 \Rightarrow x_1$ is a point of minima

$f''(x_1) < 0 \Rightarrow x_1$ is a point of maxima

Let $f(x) = x^3 - 3(7-a)x^2 - 3(9-a^2)x + 2$

46. The values of parameter 'a' if $f(x)$ has a negative point of local minimum, are
 (a) ϕ (b) $\left(-\infty, \frac{58}{14}\right)$
 (c) $(-3, 3)$ (d) none of these
47. The values of parameter 'a' if $f(x)$ has a positive point of local maxima are
 (a) ϕ (b) $(-\infty, -3) \cup \left(3, \frac{58}{14}\right)$
 (c) $\left(-\infty, \frac{58}{14}\right)$ (d) none of these
48. The values of parameter 'a' if $f(x)$ has points of extrema which are opposite in sign are
 (a) ϕ (b) $(-3, 3)$
 (c) $\left(-\infty, \frac{58}{14}\right)$ (d) none of these

Paragraph for Q.No. 49 to 51

Let $f(x) = ax^2 + bx + c$; $a, b, c \in R$

It is given that $|f(x)| \leq 1, \forall |x| \leq 1$

49. The possible value of $|a+b|$ if $4a^2 + 3b^2$ is maximum, is given by
 (a) 1 (b) 0 (c) 2 (d) 3
50. The possible value of $|a+b|$ if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by
 (a) 1 (b) 0 (c) 2 (d) 3

51. The possible maximum value of $\frac{8}{3}a^2 + 2b^2$ is given by

- (a) 32 (b) $\frac{32}{3}$ (c) $\frac{2}{3}$ (d) $\frac{16}{3}$

Matrix-Match Type

52. Match the following.

Column-I	Column-II
(A) $f(x) = x^2 \log_e x$	(p) $f(x)$ has one point of minima.
(B) $f(x) = x \log_e x$	(q) $f(x)$ has one point of maxima.
(C) $f(x) = \frac{\log_e x}{x}$	(r) $f(x)$ increases in $(0, e)$.
(D) $f(x) = x^{-x}$	(s) $f(x)$ decreases in $(0, 1/e)$.

53. Match the following.

Column-I	Column-II
(A) $f(x) = (x-1)^3(x+2)^5$	(p) has points of maxima.
(B) $f(x) = 3\sin x + 4\cos x - 5x$	(q) has points of minima.
(C) $f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0 < x \leq 1 \\ x^2 - 4x + 4, & 1 < x < 2 \end{cases}$	(r) has points of inflection.
(D) $f(x) = (x-1)^{3/5}$	(s) has no points of extrema.

54. Match the following, for the function $f(x) = ax^2 - b|x|$.

Column-I	Column-II
(A) $f(x)$ has local max. at $x = 0$	(p) When $a > 0$, $b > 0$
(B) $f(x)$ has local min. at $x = 0$	(q) When $a > 0$, $b < 0$
(C) $f(x)$ has local extremum at $x = \frac{b}{2a}$	(r) When $a < 0$, $b < 0$
(D) $f(x)$ is not diff. at $x = 0$	(s) When $a < 0$, $b > 0$

Integer Answer Type

55. From a point perpendicular tangents are drawn to ellipse $x^2 + 2y^2 = 2$. The chord of contact touches a circle which is concentric with given ellipse. Then find the ratio of maximum and minimum area of circle.

56. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x | x^2 + 20 \leq 9x\}$ is
57. If the graph of $f(x) = 2x^3 + ax^2 + bx$, where $a, b \in \mathbb{N}$ cuts the x -axis at three real and distinct points, then the minimum value of $(a^2 + b^2 - 4)$, is
58. Given a point $(2, 1)$. If the minimum perimeter of a triangle with one vertex at $(2, 1)$, one on the x -axis, and one on the line $y = x$, is k , then $[k]$ is equal to (where $[\cdot]$ denotes the greatest integer function)
59. If one root of $x^2 - 4ax + a + f(a) = 0$ is three times the other and if minimum value of $f(a)$ is α , then $|12\alpha|$ has a value
60. The minimum value of $\frac{\sec^4 \alpha}{\tan^2 \beta} + \frac{\sec^4 \beta}{\tan^2 \alpha}$, $\alpha, \beta \neq \frac{K\pi}{2}, K \in \mathbb{I}$, is
61. For a twice differentiable function $f(x)$, a function $g(x)$ is defined as $g(x) = (f'(x))^2 + f(x)f''(x)$ on $[a, e]$. If $a < b < c < d < e$ and $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, then the minimum number of roots of the equation $g(x) = 0$, is/are
62. Let $P(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \rightarrow 0} \left(1 + \frac{P(x)}{x^2}\right) = 2$, then the value of $P(2)$ is
63. In the coordinate plane, the region M consists of all points (x, y) satisfying the inequalities $y \geq 0, y \leq x$ and $y \leq 2 - x$ simultaneously. The region N which varies with parameter t , consists of all the points (x, y) satisfying the inequalities $t \leq x \leq t + 1$ and $0 \leq t \leq 1$ simultaneously. If the area of the region $M \cap N$ is a function of t , i.e., $M \cap N = f(t)$ and if α is the value of t for which this area is maximum, then the numerical value of 2α is
64. Put numbers 1, 2, 3, 4, 5, 6, 7, 8 at the vertices of a cube, such that the sum of any three numbers on any face is not less than 10. The minimum sum of the four number on a face is k , then $k/2$ is equal to

SOLUTIONS

1. (c) : $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$$\therefore \frac{dy}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \Rightarrow \frac{dy}{dx} = y - \frac{x^n}{n!}$$

2. (a)

3. (a) : $\log |x| = \begin{cases} \log x, & \text{if } x > 0 \\ \log(-x), & \text{if } x < 0 \end{cases}$

Hence, $\frac{d}{dx} \{\log |x|\} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases}$

$$\therefore \frac{d}{dx} \{\log |x|\} = \frac{1}{x}, \text{ if } x \neq 0$$

4. (d) : $y = a \sin x + b \cos x$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = a \cos x - b \sin x$$

$$\text{Now } \left(\frac{dy}{dx}\right)^2 = (a \cos x - b \sin x)^2$$

$$= a^2 \cos^2 x + b^2 \sin^2 x - 2ab \sin x \cos x$$

$$\text{and } y^2 = (a \sin x + b \cos x)^2$$

$$= a^2 \sin^2 x + b^2 \cos^2 x + 2ab \sin x \cos x$$

$$\text{So, } \left(\frac{dy}{dx}\right)^2 + y^2 = a^2(\sin^2 x + \cos^2 x) + b^2(\sin^2 x + \cos^2 x)$$

$$\text{Hence } \left(\frac{dy}{dx}\right)^2 + y^2 = (a^2 + b^2) = \text{constant}$$

5. (a) : Let $y = \tan^{-1} \sqrt{\frac{1 + \cos \frac{x}{2}}{1 - \cos \frac{x}{2}}} = \tan^{-1} \sqrt{\frac{2 \cos^2 \frac{x}{4}}{2 \sin^2 \frac{x}{4}}}$

$$y = \tan^{-1} \cot \frac{x}{4} = \tan^{-1} \tan \left(\frac{\pi}{2} - \frac{x}{4}\right) = \frac{\pi}{2} - \frac{x}{4}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{4}$$

6. (a) : $y = \log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$

$$= \log_{10} x + \frac{\log_e 10}{\log_e x} + 1 + 1 \Rightarrow \frac{dy}{dx} = \frac{1}{x} \log_{10} e - \frac{\log_e 10}{x(\log_e x)^2}$$

7. (a) : $y = \log \left(\frac{1+x}{1-x}\right)^{1/4} - \frac{1}{2} \tan^{-1} x$

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \left(\frac{1-x}{1+x}\right)^{1/4} \frac{1}{4} \left(\frac{1+x}{1-x}\right)^{-3/4} \left[\frac{(1-x) + (1+x)}{(1-x)^2}\right] - \frac{1}{2} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{2} \left(\frac{1-x}{1+x} \right) \frac{1}{(1-x)^2} - \frac{1}{2(1+x^2)}$$

$$= \frac{1}{2} \cdot \frac{1}{(1-x^2)} - \frac{1}{2} \frac{1}{(1+x^2)} = \frac{x^2}{1-x^4}$$

8. (c) : $y = \tan^{-1} \frac{4x}{1+5x^2} + \tan^{-1} \frac{2+3x}{3-2x}$

$$= \tan^{-1} \frac{5x-x}{1+5x \cdot x} + \tan^{-1} \frac{\frac{2}{3}+x}{1-\frac{2}{3} \cdot x}$$

$$= \tan^{-1} 5x - \tan^{-1} x + \tan^{-1} \frac{2}{3} + \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{5}{1+25x^2}$$

9. (b) : $f(x) = \sqrt{1+\cos^2(x^2)}$

$$f'(x) = \frac{1}{2\sqrt{1+\cos^2(x^2)}} \cdot (2\cos x^2) \cdot (-\sin x^2) \cdot (2x)$$

$$f'(x) = \frac{-x \sin 2x^2}{\sqrt{1+\cos^2(x^2)}}$$

At $x = \frac{\sqrt{\pi}}{2}$, $f'\left(\frac{\sqrt{\pi}}{2}\right) = \frac{-\frac{\sqrt{\pi}}{2} \cdot \sin \frac{2\pi}{4}}{\sqrt{1+\cos^2 \frac{\pi}{4}}} = \frac{-\frac{\sqrt{\pi}}{2} \cdot 1}{\sqrt{\frac{3}{2}}}$

$\therefore f'\left(\frac{\sqrt{\pi}}{2}\right) = -\sqrt{\frac{\pi}{6}}$

10. (c) : $\frac{d}{dx} [(1+x^2) \tan^{-1} x] = 1 + 2x \tan^{-1} x$

11. (b) : $\frac{d}{dx} \log(\sqrt{x-a} + \sqrt{x-b})$

$$= \left(\frac{1}{\sqrt{x-a} + \sqrt{x-b}} \right) \frac{1}{2} \left[\frac{1}{\sqrt{x-a}} + \frac{1}{\sqrt{x-b}} \right]$$

$$= \left[\frac{\sqrt{x-a} + \sqrt{x-b}}{\sqrt{x-a} + \sqrt{x-b}} \right] \frac{1}{2\sqrt{(x-a)(x-b)}} = \frac{1}{2\sqrt{(x-a)(x-b)}}$$

12. (b) : $\frac{d}{dx} \tan^{-1} \frac{4\sqrt{x}}{1-4x}$

$$= \frac{1}{1 + \left(\frac{4\sqrt{x}}{1-4x} \right)^2} \cdot \left[\frac{(1-4x)4 \left(\frac{1}{2\sqrt{x}} \right) - 4\sqrt{x}(-4)}{(1-4x)^2} \right]$$

$$= \frac{2(1+4x)}{\sqrt{x}(1+4x)^2} = \frac{2}{\sqrt{x}(1+4x)}$$

13. (a) : $y = \sec^{-1} \left(\frac{\sqrt{x}+1}{\sqrt{x}-1} \right) + \sin^{-1} \left(\frac{\sqrt{x}-1}{\sqrt{x}+1} \right)$

$$= \cos^{-1} \left(\frac{\sqrt{x}-1}{\sqrt{x}+1} \right) + \sin^{-1} \left(\frac{\sqrt{x}-1}{\sqrt{x}+1} \right) = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$$

14. (a) : $y = \sqrt{\left(\frac{1+\tan x}{1-\tan x} \right)}$ or $y = \sqrt{\tan \left(\frac{\pi}{4} + x \right)}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan \left(\frac{\pi}{4} + x \right)}} \sec^2 \left(\frac{\pi}{4} + x \right)$$

$$= \frac{1}{2} \sqrt{\left[\frac{1-\tan x}{1+\tan x} \right]} \sec^2 \left(\frac{\pi}{4} + x \right)$$

15. (d) : Taking log, $\log y = x + \log \log x - 2 \log x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \frac{1}{x \log x} - \frac{2}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x \log x}{x^2} \left[\frac{x \log x + 1 - 2 \log x}{x \log x} \right]$$

$$= \frac{e^x [(x-2) \log x + 1]}{x^3}$$

16. (d) : $r = \left[2\phi + \cos^2 \left(2\phi + \frac{\pi}{4} \right) \right]^{1/2}$

$$\Rightarrow \frac{dr}{d\phi} = \frac{1}{2} \left[2\phi + \cos^2 \left(2\phi + \frac{\pi}{4} \right) \right]^{-1/2}$$

$$\left[2 - 2 \times 2 \sin \left(2\phi + \frac{\pi}{4} \right) \times \cos \left(2\phi + \frac{\pi}{4} \right) \right]$$

$$\left(\frac{dr}{d\phi} \right)_{\phi=\frac{\pi}{4}} = \frac{1}{2} \left[\frac{\pi}{2} + \cos^2 \frac{3\pi}{4} \right]^{-1/2} \times 2 \left[\left(1 - \sin \left(\pi + \frac{\pi}{2} \right) \right) \right]$$

$$\Rightarrow \left(\frac{dr}{d\phi} \right)_{\phi=\frac{\pi}{4}} = \frac{1}{2} \left(\frac{\pi}{2} + \frac{1}{2} \right)^{-1/2} \times 2(1+1) = 2 \times \left(\frac{2}{\pi+1} \right)^{1/2}$$

17. (b)

18. (b) : $\frac{d}{dx} (\log \tan x) = \frac{1}{\tan x} \sec^2 x = \frac{\cos x}{\cos^2 x \sin x}$

$$= \frac{2}{2 \cos x \sin x} = 2 \operatorname{cosec} 2x$$

19. (a) : $\frac{d}{dx} \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 = \frac{d}{dx} \left[x + \frac{1}{x} + 2 \right] = 1 - \frac{1}{x^2}$

20. (c) : $y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

From given options we have,

$$x^2 \cdot \frac{dy}{dx} - xy + 2 = x^2 \left(1 - \frac{1}{x^2}\right) - x \left(x + \frac{1}{x}\right) + 2 = 0$$

21. (a) : $\frac{d}{dx}(e^x \log \sin 2x) = e^x \log \sin 2x + 2e^x \frac{1}{\sin 2x} \cos 2x$
 $= e^x \log \sin 2x + e^x 2 \cot 2x = e^x (\log \sin 2x + 2 \cot 2x)$

22. (c) : Put $x = \cos \theta$

$$\therefore y = \tan^{-1} \frac{\cos \theta}{1 + \sin \theta} + \sin \left[2 \tan^{-1} \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \right]$$

$$= \tan^{-1} \frac{\sin \phi}{1 + \cos \phi} + \sin \left[2 \tan^{-1} \tan \left(\frac{\theta}{2} \right) \right],$$

{where $\phi = 90^\circ - \theta$ }

$$= \tan^{-1} \tan \left(\frac{\phi}{2} \right) + \sin \left(2 \cdot \frac{\theta}{2} \right) = \left(\frac{\phi}{2} \right) + \sin \theta$$

$$= \frac{\pi}{4} - \frac{\theta}{2} + \sqrt{1 - \cos^2 \theta} = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x + \sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{1 - x^2}} + \frac{1}{2\sqrt{1 - x^2}} (-2x) = \frac{1 - 2x}{2\sqrt{1 - x^2}}$$

23. (b)

24. (a) : $y = \tan^{-1} \left(\frac{x}{1 + \sqrt{1 - x^2}} \right)$

Put $x = \sin \theta$

$$\therefore y = \tan^{-1} \left(\frac{\sin \theta}{1 + \sqrt{1 - \sin^2 \theta}} \right) = \tan^{-1} \left(\frac{\sin \theta}{1 + \cos \theta} \right)$$

$$= \tan^{-1} \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \tan^{-1} \tan \frac{\theta}{2} = \frac{\theta}{2}$$

So, $y = \frac{\sin^{-1} x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1 - x^2}}$

25. (b) : $3f(x) - 2f(1/x) = x$
 Let $1/x = y$, then $3f(1/y) - 2f(y) = 1/y$

$$\Rightarrow -2f(y) + 3f(1/y) = 1/y$$

$$\Rightarrow -2f(x) + 3f(1/x) = 1/x$$

From $3 \times (i) + 2 \times (ii)$, we have

$$9f(x) - 6f(1/x) - 4f(x) + 6f(1/x) = 3x + 2/x$$

$$\Rightarrow 5f(x) = 3x + \frac{2}{x} \Rightarrow f(x) = \frac{1}{5} \left[3x + \frac{2}{x} \right]$$

$$\Rightarrow f'(x) = \frac{1}{5} \left[3 - \frac{2}{x^2} \right] \Rightarrow f'(2) = \frac{1}{5} \left[3 - \frac{2}{4} \right] = \frac{1}{2}$$

26. (c)

27. (c) : $y = A \cos(nx) + B \sin(nx)$

$$\therefore dy/dx = -nA \sin(nx) + nB \cos(nx)$$

Again $\frac{d^2 y}{dx^2} = -n^2 A \cos(nx) - n^2 B \sin(nx)$

$$= -n^2 [A \cos(nx) + B \sin(nx)] \Rightarrow \frac{d^2 y}{dx^2} = -n^2 y$$

28. (b) : $f(x) = a \sin(\log x)$

Differentiating w.r.t. x , we get

$$f'(x) = a \cos(\log x) \frac{1}{x}$$

Again $f''(x) = -\frac{1}{x^2} a \cos(\log x) - \frac{1}{x^2} a \sin(\log x)$

$$\Rightarrow x^2 f''(x) = -[a \cos(\log x) + a \sin(\log x)]$$

$$\text{Now } x^2 f''(x) + x f'(x) = -a \sin(\log x) = -f(x)$$

29. (c) : It is a fundamental property, which follows from generalized derivative test.

30. (c) : $f(x) = 2x^3 - 21x^2 + 36x - 30$

$$\Rightarrow f'(x) = 6x^2 - 42x + 36 = 6(x - 6)(x - 1)$$

$$\therefore f'(x) = 0 \Rightarrow x = 6, 1 \text{ and } f''(x) = 12x - 42$$

$$\text{Here } f''(1) = -30 \text{ and } f''(6) = 30$$

Hence $f(x)$ has maxima at $x = 1$ and minima at $x = 6$.

31. (a) : Let $f(x) = x + \frac{1}{x}$

Differentiating with respect to x , we get

$$f'(x) = 1 - \frac{1}{x^2} \leq 0 \Rightarrow 1 \leq \frac{1}{x^2} \text{ or } x^2 \leq 1$$

Hence $x \in [-1, 1]$.

32. (c) : Let $f(x) = \sin x - bx + c$

$$\therefore f'(x) = \cos x - b > 0 \text{ or } \cos x > b \text{ or } b < -1.$$

33. (b) : We have, $f'(x) = \cos x + \sin x$

Now $f(x)$ is increasing function of x , if

$$f'(x) = \cos x + \sin x > 0 \text{ or } \sqrt{2} \cos \left(x - \frac{\pi}{4} \right) > 0$$

...(i) $\Rightarrow 0 \leq x < \frac{3\pi}{4} \text{ i.e. } f'(x) > 0 \text{ in } \left[0, \frac{3\pi}{4} \right)$

34. (a) : We have, $f(x) = 5^{-x}$

...(ii) $\Rightarrow f'(x) = -5^{-x} \log_e 5 = -\frac{\log_e 5}{5^x} \Rightarrow f'(x) < 0 \text{ for all } x$

i.e., $f(x)$ is decreasing for all x .

35. (b) : Since $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing for all real values of x

$$\therefore f'(x) < 0 \forall x$$

$$\Rightarrow \sqrt{3} \cos x + \sin x - 2a < 0 \forall x$$

$$\Rightarrow \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x < a \forall x$$

$$\Rightarrow \sin\left(x + \frac{\pi}{3}\right) < a \forall x \Rightarrow a \geq 1 \left[\because \sin\left(x + \frac{\pi}{3}\right) \leq 1 \right]$$

36. (a, b, c, d) : $f(x) = \left(\frac{a^2-1}{3}\right)x^3 + (a-1)x^2 + 2x + 1$

$$f'(x) > 0 \Rightarrow \frac{(a^2-1)3x^2}{3} + 2x(a-1) + 2 > 0$$

$$\Rightarrow (a^2-1)x^2 + 2x(a-1) + 2 > 0$$

$$\Rightarrow 4(a-1)^2 - 8(a^2-1) < 0 \quad \{\because b^2 - 4ac < 0\}$$

$$\Rightarrow (a-1)^2 - 2(a^2-1) < 0 \Rightarrow (a+3)(a-1) > 0$$

$$\Rightarrow a \in (-\infty, -3) \cup (1, \infty)$$

37. (a, c) : Let $f(x) = x \tan x \Rightarrow f'(x) = \tan x + x \sec^2 x > 0$

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(c) > 0 \Rightarrow f(\beta) > f(\alpha) \Rightarrow \frac{\tan \beta}{\tan \alpha} > \frac{\alpha}{\beta}$$

Now, let $f(x) = \log(1+x) \Rightarrow f'(x) = \frac{1}{1+x}$

Now, let $f(x) = \log(1+x) \Rightarrow f'(x) = \frac{1}{1+x}$

$$0 < c < x \Rightarrow 1 < 1+c < 1+x \Rightarrow \frac{1}{1+x} < \frac{1}{1+c} < 1$$

$$\Rightarrow \frac{x}{1+x} < \frac{x}{1+c} < x \Rightarrow \frac{x}{1+x} < \log(1+x) < x$$

Equality holds good for $x = 0$

38. (b, c, d) : $f(x) = \frac{e^x}{1+x^2}$, $g(x) = f'(x) = \frac{(x-1)^2 e^x}{(1+x^2)^2}$,

$$g'(x) = \frac{(x-1)(x^3 - 3x^2 + 5x + 1)}{(x^2+1)^3} e^x$$

Let $h(x) = x^3 - 3x^2 + 5x + 1$, $h'(x) = 3x^2 - 6x + 5$, $D < 0$ so $h(x)$ has only one real roots. Also $g'(-1) g'(0) < 0$. So the root $\in (-1, 0)$. Clearly $g(x)$ has two points of extremum. Maxima at $x \in (-1, 0)$ and minima at $x = 1$.

39. (a, b, d) : Given $f(x) = x^2 \cdot e^{-x^2}$

$$f'(x) = 2x \cdot e^{-x^2} + x^2 \cdot e^{-x^2}(-2x) = 2xe^{-x^2}[1 - x^2]$$

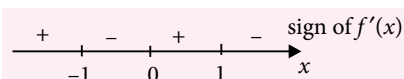
$f(x)$ has local maxima at $x = -1$ and 1

$f(x)$ has local minima at $x = 0$

Now $f(0) = 0$

$$f(1) = \frac{1}{e} \text{ and as } x \rightarrow \infty, f(x) \rightarrow 0$$

So range of $f(x)$ is $\left[0, \frac{1}{e}\right]$



40. (a, c, d) : When $f(x)$ is continuous at $x = 2$ and $f'(x)$ changes sign from +ve to -ve.

$\Rightarrow f(x)$ attains max.

At $x = 2$ if

$$\frac{k^3(k-1)^2}{k^2-k-2} = 0 \Rightarrow k = 0, 1$$

When $f(x)$ is discontinuous at $x = 2$, $f'(x)$ changes its sign from +ve to -ve $f(x)$ will attain maximum if

$$\lim_{x \rightarrow 2^+} f(x) < f(2) \text{ and } \lim_{x \rightarrow 2^-} f(x) = f(2)$$

i.e. if $k \in (-\infty, -1) \cup (0, 1) \cup (1, 2)$

$$\Rightarrow k \in (-\infty, -1) \cup [0, 2)$$

41. (b, c, d) : The answers follow from definition.

42. (a, b, d) : (a) Let $f(x) = \ln(1+x) - x + \frac{x^2}{2}$

$$f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x}$$

$$f(0) = 0, f'(x) > 0 \forall x \in (0, \infty) \Rightarrow f(x) > 0$$

(b) Let $f(x) = \ln(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{1}{(x+1)^2} = \frac{x}{(1+x)^2}$$

$$f(0) = 0, f'(x) > 0 \forall x \in (0, \infty) \Rightarrow f(x) > 0$$

(c) Let $y = \cos x$, $y = -\frac{x}{4}$

by graph it is clear that

$$\cos x > \frac{-x}{4} \text{ is not true } \forall x \in (0, \infty)$$

(d) Let $f(x) = x + 1 - 2 \tan^{-1} x$

$$f'(x) = \frac{x^2-1}{1+x^2} = \frac{(x-1)(x+1)}{x^2+1}, f(0) = 1$$

\therefore At $x = 1$

$$f(1) = 2 \left(1 - \frac{\pi}{4}\right) > 0 \Rightarrow f(x) > 0 \forall x \in (0, \infty)$$

43. (b, c, d) : Solving we get $x = 2^{1/2}$, $y = 3^{1/3}$, $z = 5^{1/5}$

Using graph of $x^{1/x}$, we get

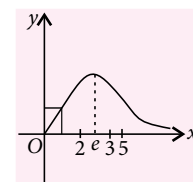
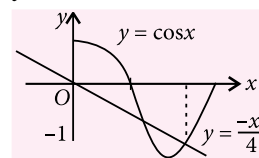
$$3^{1/3} > 5^{1/5}$$

$$\text{Also, } 2^{1/2} < 3^{1/3} \quad [\because 2^3 < 3^2]$$

$$2^{1/2} > 5^{1/5} \quad [\because 2^5 > 5^2]$$

$$\Rightarrow y > x > z$$

Hence (b), (c), & (d) are not true.



44. (b, c, d) : $f(x) = x \cos \frac{1}{x}, x \geq 1$

$\therefore f'(x) = \cos \frac{1}{x} + \frac{1}{x} \sin \frac{1}{x}$

45. (d) : $g'(x) = f'((\tan x - 1)^2 + 3) \cdot (2 \tan x - 2) \sec^2 x$

Now, $f''(x) > 0 \Rightarrow f'(x)$ is increasing.

$\Rightarrow f'((\tan x - 1)^2 + 3) > f'(3) = 0 \forall x \in (0, \pi/4) \cup (\pi/4, \pi/2)$

Also, $(\tan x - 1) > 0 \forall x \in (\pi/4, \pi/2)$

$\therefore g(x)$ is increasing in $(\pi/4, \pi/2)$.

(46-48) :

We have $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$

$f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2)$

For real roots $D \geq 0, a \leq \frac{58}{14}$... (i)

Let $f'(x)$ has two roots x_1 and x_2 ($x_2 > x_1$)

Minima at $x = x_2$

46. (a) : Both roots -ve $\Rightarrow 2(7 - a) < 0 \Rightarrow a > 7$

Not possible

47 (b) : Both roots are +ve \Rightarrow Sum of roots $> 0 \Rightarrow a < 7$

... (ii)

Product of roots $> 0 \Rightarrow a \in (-\infty, -3) \cup (3, \infty)$... (iii)

From (i), (ii), (iii), we get $a \in (-\infty, -3) \cup \left(3, \frac{58}{14}\right)$

48. (b) : For points of opposite sign,

Product of roots < 0

$a \in (-3, 3)$

(49-51) :

49. (c) 50. (c) 51. (b)

Now $|f(1) - f(0)| \leq 2 \Rightarrow |a + b| \leq 2 \Rightarrow (a + b)^2 \leq 4$

$|f(-1) - f(0)| \leq 2 \Rightarrow |a - b| \leq 2 \Rightarrow (a - b)^2 \leq 4$

Now, $4a^2 + 3b^2 = 2(a + b)^2 + 2(a - b)^2 - b^2 \leq 16 - b^2$

$(4a^2 + 3b^2)_{\max} = 16$ when $b = 0$

$\Rightarrow |a + b| = |a - b| = |a| = 2$

Also the possible ordered triplet (a, b, c) are $(2, 0, -1)$

or $(-2, 0, 1)$

Also $\frac{8}{3}a^2 + 2b^2 = \frac{2}{3}(4a^2 + 3b^2) \leq \frac{2}{3} \times 16 \leq \frac{32}{3}$

52. (A) \rightarrow (p), (B) \rightarrow (p, s), (C) \rightarrow (q, r), (D) \rightarrow (q)

(A) We have, $f(x) = x^2 \log_e x$

\therefore For $f'(x) = x(2 \log x + 1) = 0, \Rightarrow x = \frac{1}{\sqrt{e}}$

Which is the point of minima as derivative changes sign from negative to positive.

Also, the function decreases in $\left(0, \frac{1}{\sqrt{e}}\right)$

(B) Given, $f(x) = x \log_e x$

$\therefore f'(x) = x \times \frac{1}{x} + \log x \times 1 = 1 + \log x$ and $f''(x) = \frac{1}{x}$

For $f'(x) = 0 \Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$

$f''(x) = \frac{1}{1/e} = e > 0$ at $x = \frac{1}{e}$

$\Rightarrow f(x)$ is min for $x = \frac{1}{e}$

(C) We have, $f(x) = \frac{\log x}{x}$

$\therefore f'(x) = \frac{1 - \log x}{x^2} = 0, x = e$. Also, derivative changes

sign from positive to negative at $x = e$, hence it is the point of maxima.

(D) We have, $f(x) = x^{-x}$

$\therefore f'(x) = -x^{-x}(1 + \log x) = 0 \Rightarrow x = \frac{1}{e}$

Which is clearly point of maxima.

53. (A) \rightarrow (q, r), (B) \rightarrow (r, s), (C) \rightarrow (p, r), (D) \rightarrow (r, s)

(A) $f(x) = (x - 1)^3(x + 2)^5$

$f'(x) = (x - 1)^2(x + 2)^4(8x + 1) = 0$

$\Rightarrow x = 1, -2, -1/8$

\Rightarrow point of minima at $x = -\frac{1}{8}$ and

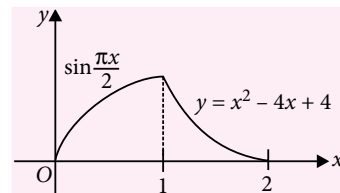
Points of inflection are $x = 1, -2$

(B) $f(x) = 3 \sin x + 4 \cos x - 5x$

$f'(x) = 3 \cos x - 4 \sin x - 5 \leq 0$

$f''(x) = -3 \sin x - 4 \cos x = 0$ for infinite value of x

(C)



$x = 1$ point of maxima as well as point of inflection

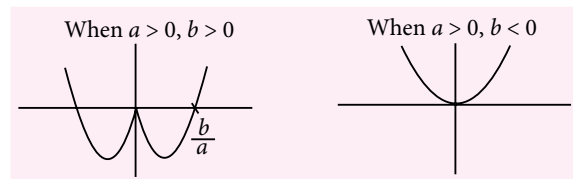
(D) $f'(x) = \frac{3}{5}(x - 1)^{-2/5} \geq 0 \forall x \in R$

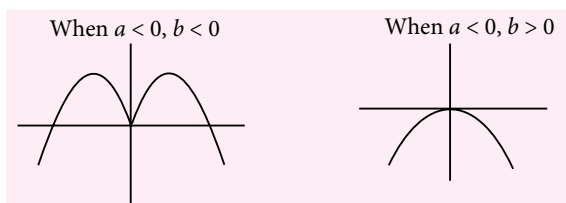
$f''(x) = \frac{-3}{5} \times \frac{2}{5}(x - 1)^{-7/5}$

At $x = 1$ $f'(x)$ & $f''(x) = 0 \therefore x = 1$ is a point of inflection

54. (A) \rightarrow (p, s), (B) \rightarrow (q, r), (C) \rightarrow (p, r), (D) \rightarrow (p, q, r, s)

The graphs for various cases are given as below





55. (4) : The director circle of ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is $x^2 + y^2 = 3$

Let a point $P(\sqrt{3} \cos \theta, \sqrt{3} \sin \theta)$

\therefore Equation of chord of contact is

$$x \cdot \sqrt{3} \cos \theta + 2y \cdot \sqrt{3} \sin \theta - 2 = 0$$

It touches $x^2 + y^2 = r^2$

$$\therefore r = \frac{2}{\sqrt{3 \cos^2 \theta + 12 \sin^2 \theta}} = \frac{2}{\sqrt{3 + 9 \sin^2 \theta}}$$

$$r_{\max} = \frac{2}{\sqrt{3}} \text{ and } r_{\min} = \frac{2}{\sqrt{12}} \Rightarrow \frac{A_{\max}}{A_{\min}} = 4$$

56. (7) : The given function is $f(x) = 2x^3 - 15x^2 + 36x - 48$ and $A = \{x \mid x^2 + 20 \leq 9x\}$

$$\Rightarrow A = \{x \mid x^2 - 9x + 20 \leq 0\}$$

$$\Rightarrow A = \{x \mid (x-4)(x-5) \leq 0\} \Rightarrow A = [4, 5]$$

$$\text{Also } f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$$

$$= 6(x-2)(x-3)$$

Clearly $\forall x \in A, f'(x) > 0$

$\therefore f$ is strictly increasing function on A .

\therefore Maximum value of f on A is

$$f(5) = 2 \times 5^3 - 15 \times 5^2 + 36 \times 5 - 48$$

$$= 250 - 375 + 180 - 48 = 7$$

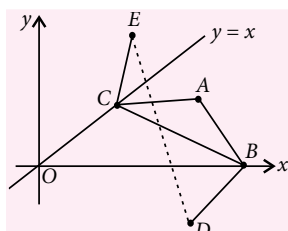
$$\mathbf{57. (6) :} f'(x) = 6x^2 + 2ax + b$$

Since, $f(x)$ has three real and distinct roots, so $\Rightarrow 4a^2 - 24b \geq 0$

$$\Rightarrow a^2 \geq 6b \Rightarrow a \geq 3, b \geq 1 \Rightarrow a = 3, b = 1$$

$$\therefore a^2 + b^2 - 4 = 9 + 1 - 4 = 6$$

58. (3) : Let $D = (2, -1)$ be the reflection of A in x -axis, and let $E = (1, 2)$ be the reflection in the line $y = x$. Then $AB = BD$ and $AC = CE$, so the perimeter of $\triangle ABC$ is $DB + BC + CE \geq DE = \sqrt{1+9} = \sqrt{10}$



59. (1) : Let θ and 3θ be the roots of given equation.

Then $4\theta = 4a \Rightarrow \theta = a$

$$\text{and } a - 3a^2 + f(a) = 0$$

$$\Rightarrow f(a) = 3a^2 - a \Rightarrow f_{\min} \text{ is } \frac{-1}{12}$$

$$\mathbf{60. (8) :} \frac{(a+1)^2}{b} + \frac{(b+1)^2}{a} = \frac{a^2}{b} + \frac{1}{b} + \frac{b^2}{a} + \frac{1}{a} + 2\left(\frac{a}{b} + \frac{b}{a}\right) \\ \geq 4 \left[\frac{a^2}{b} \cdot \frac{1}{b} \cdot \frac{b^2}{a} \cdot \frac{1}{a} \right]^{1/4} + 4 \left(\frac{a}{b} \cdot \frac{b}{a} \right)^{1/2} \geq 8$$

Where $a = \tan^2 \alpha, b = \tan^2 \beta$

61. (6) : $\because f(b)f(c) < 0$ and $f(c)f(d) < 0$

$\Rightarrow f(x) = 0$ has at least four roots, a, c_1, c_2, e , where $c_1 \in (b, c)$ and $c_2 \in (c, d)$. Then, by Rolle's theorem, $f'(x) = 0$ has at least three roots in, $(a, c_1), (c_1, c_2), (c_2, e)$. So $f(x)f'(x) = 0$ has at least 7 roots.

$$\therefore g(x) = \frac{d}{dx} \{f(x)f'(x)\} = 0 \text{ has at least 6 roots.}$$

62. (0) : Let $P(x) = a_0x^4 + \dots + a_4$ by hypothesis, $P(1) = 0$ and $P'(2) = 0$

$$\Rightarrow 4a_0 + 3a_1 + 2a_2 + a_3 = 0$$

$$\text{and } 32a_0 + 12a_1 + 4a_2 + a_3 = 0$$

$$\text{Also, } \lim_{x \rightarrow 0} \frac{P(x)}{x^2} = 1 \Rightarrow a_4 = 0 \text{ and } a_3 = 0 \text{ hence}$$

$$\lim_{x \rightarrow 0} (a_0x^2 + a_1x + a_2) = 1 \Rightarrow a_2 = 1$$

$$\text{Solving, we get } a_0 = \frac{1}{4}, a_1 = -1, a_2 = 1, a_3 = 0, a_4 = 0$$

$$\therefore P(x) = \frac{1}{4}x^4 - x^3 + x^2 \Rightarrow P(2) = 0$$

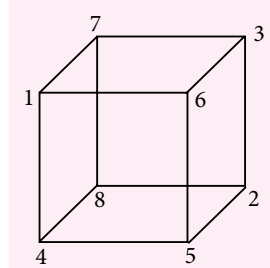
$$\mathbf{63. (1) :} M \cap N = f(t) = -t^2 + t + 1/2 = \frac{3}{4} - \left(t - \frac{1}{2}\right)^2 \\ f(t) \text{ is maximum for } t = 1/2 \\ \text{i.e. } \alpha = \frac{1}{2} \Rightarrow 2\alpha = 1$$

64. (8) : Suppose that the four numbers on a face of the cube is a_1, a_2, a_3, a_4 such that their sum reaches the minimum and $a_1 < a_2 < a_3 < a_4$. Since the maximum sum of any three numbers less than 5 is 9, we have $a_4 \geq 6$ and $a_1 + a_2 + a_3 + a_4 \geq 16$.

As seen in figure, we have

$$2 + 3 + 5 + 6 = 16 \text{ and}$$

that means minimum sum of four numbers on a face is 16.



ACE YOUR WAY CBSE

Vector Algebra

IMPORTANT FORMULAE

- ▶ Position vector of a point $P(x, y, z)$ is $\vec{OP}(=\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$, and its magnitude by $\sqrt{x^2 + y^2 + z^2}$.
- ▶ Relation between the magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of any vector is $l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$
- ▶ The vector sum of the three sides of a triangle taken in order is $\vec{0}$.
- ▶ The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- ▶ The vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of \vec{a} .
- ▶ The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are \vec{a} and \vec{b} respectively, in the ratio $m : n$
 - ▶ internally, is given by $\frac{n\vec{a} + m\vec{b}}{m + n}$.
 - ▶ externally, is given by $\frac{m\vec{b} - n\vec{a}}{m - n}$.
- ▶ The scalar product of two given vectors \vec{a} and \vec{b} having angle θ between them is $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$
 - ▶ $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
 - ▶ If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$
 - ▶ If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -|\vec{a}| |\vec{b}|$
- ▶ $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
- ▶ Projection of a vector \vec{a} on other vector \vec{b} , is $\vec{a} \cdot \left(\frac{\vec{b}}{|\vec{b}|} \right)$ or $\frac{1}{|\vec{b}|} (\vec{a} \cdot \vec{b})$.
 - ▶ If $\theta = 0$, then the projection vector of \vec{AB} will be \vec{AB} itself and if $\theta = \pi$, then the projection vector of \vec{AB} will be \vec{BA} .
 - ▶ If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of \vec{AB} will be zero vector.
- ▶ If θ is the angle between two vectors \vec{a} and \vec{b} , then their cross product is given by $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$ where \hat{n} is a unit vector perpendicular to the plane containing \vec{a} and \vec{b} such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system of coordinate axes.
 - ▶ $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$

► If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}|$.

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$$

► If we have two vectors \vec{a} and \vec{b} , given in component forms $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ any scalar, then

► $\vec{a} \pm \vec{b} = (a_1 \pm b_1)\hat{i} + (a_2 \pm b_2)\hat{j} + (a_3 \pm b_3)\hat{k}$

► $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

► $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

► $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ or $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

► If \vec{a} and \vec{b} represent the adjacent sides of a triangle, then its area is given as $\frac{1}{2} |\vec{a} \times \vec{b}|$

► Scalar Triple product : For three vectors \vec{a}, \vec{b} and \vec{c} the scalar $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product of \vec{a}, \vec{b} and \vec{c} and is denoted by $[\vec{a} \vec{b} \vec{c}]$. Thus, $[\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$

► Coplanarity : Three vectors \vec{a}, \vec{b} and \vec{c} are coplanar iff $[\vec{a} \vec{b} \vec{c}] = 0$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. If the position vector \vec{a} of a point $(12, n)$ is such that $|\vec{a}| = 13$, find the value of n .
2. A vector \vec{r} is inclined at equal angles to OX, OY and OZ . If the magnitude of \vec{r} is 6 units, find \vec{r} .
3. Find the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other.
4. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$, it is being given that $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.
5. If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$, find $\vec{a} \times \vec{b}$ and $|\vec{a} \times \vec{b}|$.

SHORT ANSWER TYPE

6. If A, B, C have position vectors $(2, 0, 0), (0, 1, 0), (0, 0, 2)$, show that $\triangle ABC$ is isosceles.
7. Prove by vector method that the area of $\triangle ABC$ is $\frac{a^2 \sin B \sin C}{2 \sin A}$.
8. Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}, \vec{a} - 2\vec{b} + 3\vec{c}, 3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.
9. If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}, \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$, then prove that $\vec{b} = \vec{c}$.
10. Find the position vector of a point A in space such that \overrightarrow{OA} is inclined at 60° to OX and at 45° to OY and $|\overrightarrow{OA}| = 10$ units.

LONG ANSWER TYPE - I

11. Prove by vector method that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.
12. Given, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$. Find a unit vector in the direction of resultant of these vectors. Also find a vector \vec{r} which is normal to both \vec{a} and \vec{b} . What is the inclination of \vec{r} to \vec{c} ?
13. AD, BE and CF are the medians of a triangle ABC intersecting in G , show that $ar(\triangle AGB) = ar(\triangle BGC) = ar(\triangle CGA) = \frac{1}{3} ar(\triangle ABC)$.
14. If \vec{a} and \vec{b} are two vectors, show that
(i) $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$
(ii) $(\vec{a} \times \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$
15. Show that the area of the parallelogram having diagonals $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$ is $5\sqrt{3}$.

LONG ANSWER TYPE - II

16. If $\vec{a}, \vec{b}, \vec{c}$, are non-coplanar vectors, then prove that $\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}$ are non-coplanar. Is this true for $\vec{a} - \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}$?
17. The lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.
18. If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}, \vec{c} = \hat{i} - 2\hat{j} + 2\hat{k}$, find $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ and hence show that $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

19. The position vectors of the points A, B, C are respectively $(1, 1, 1)$, $(1, -1, 2)$ and $(0, 2, -1)$. Find a unit vector parallel to the plane determined by A, B, C and perpendicular to the vector $(1, 0, 1)$.

20. If \vec{a}, \vec{b} are two non-collinear vectors, show that the points having position vectors $l_1\vec{a} + m_1\vec{b}, l_2\vec{a} + m_2\vec{b}$

$$\text{and } l_3\vec{a} + m_3\vec{b} \text{ are collinear, if } \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0.$$

SOLUTIONS

1. The position vector of the point $(12, n)$ is $12\hat{i} + n\hat{j}$.

$$\therefore \vec{a} = 12\hat{i} + n\hat{j} \Rightarrow |\vec{a}| = \sqrt{12^2 + n^2}$$

$$\text{But, } |\vec{a}| = 13 \therefore 13 = \sqrt{12^2 + n^2}$$

$$\Rightarrow 169 = 144 + n^2 \Rightarrow n^2 = 25 \Rightarrow n = \pm 5$$

2. Suppose \vec{r} makes an angle α with each of the axes OX, OY and OZ. Then, its direction cosines are

$$l = \cos \alpha, m = \cos \alpha, n = \cos \alpha \Rightarrow l = m = n.$$

$$\text{Now, } l^2 + m^2 + n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \pm \frac{1}{\sqrt{3}}$$

$$\therefore \vec{r} = |\vec{r}|(\hat{i}l + \hat{j}m + \hat{k}n)$$

$$\Rightarrow \vec{r} = 6\left(\pm \frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k}\right) = 2\sqrt{3}(\pm \hat{i} \pm \hat{j} \pm \hat{k}).$$

3. If the vectors \vec{a} and \vec{b} are perpendicular to each other, then $\vec{a} \cdot \vec{b} = 0$

$$\Rightarrow (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$

$$\Rightarrow (2)(1) + \lambda(-2) + (1)(3) = 0$$

$$\Rightarrow -2\lambda + 5 = 0 \Rightarrow \lambda = \frac{5}{2}.$$

4. Given $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$$\text{Now, } (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{d} \times \vec{b}) - (\vec{a} \times \vec{c}) + (\vec{d} \times \vec{c})$$

[By distributive law]

$$= (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{d}) - (\vec{a} \times \vec{c}) - (\vec{c} \times \vec{d}) = 0$$

$$\Rightarrow (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0} \Rightarrow (\vec{a} - \vec{d}) \parallel (\vec{b} - \vec{c})$$

[Here $|\vec{a} - \vec{d}| \neq 0$ and $|\vec{b} - \vec{c}| \neq 0$ as $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$].

5. Given, $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= (9 + 2)\hat{i} - (6 + 1)\hat{j} + (4 - 3)\hat{k} = 11\hat{i} - 7\hat{j} + \hat{k}.$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{171}$$

6. We have, $\vec{AB} = \text{P.V. of B} - \text{P.V. of A}$

$$\Rightarrow \vec{AB} = (0\hat{i} + \hat{j} + 0\hat{k}) - (2\hat{i} + 0\hat{j} + 0\hat{k}) = -2\hat{i} + \hat{j} + 0\hat{k}$$

$$\therefore AB = |\vec{AB}| = \sqrt{(-2)^2 + 1^2 + 0^2} = \sqrt{5}$$

$$\vec{BC} = \text{P.V. of C} - \text{P.V. of B}$$

$$\Rightarrow \vec{BC} = (0\hat{i} + 0\hat{j} + 2\hat{k}) - (0\hat{i} + \hat{j} + 0\hat{k}) = 0\hat{i} - \hat{j} + 2\hat{k}$$

$$\therefore BC = |\vec{BC}| = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$$

Clearly, $AB = BC$. Hence, $\triangle ABC$ is isosceles.

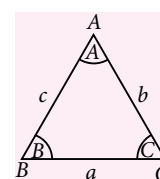
7. Area of the triangle ABC

$$= \frac{1}{2} |\vec{BC} \times \vec{BA}| = \frac{1}{2} |ac \sin B \hat{n}|$$

$$= \frac{1}{2} ac \sin B = \frac{1}{2} a \frac{c}{\sin C} \sin B \sin C$$

$$= \frac{1}{2} a \frac{a}{\sin A} \sin B \sin C$$

$$= \frac{a^2 \sin B \sin C}{2 \sin A}$$



[By sine formula]

8. Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors \vec{PQ} , \vec{PR} and \vec{PS} are coplanar. These vectors are coplanar if one of them can be expressed as a linear combination of other two. So, let $\vec{PQ} = x\vec{PR} + y\vec{PS}$

$$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = x(\vec{a} + \vec{b} - \vec{c}) + y(-\vec{a} - 9\vec{b} + 7\vec{c})$$

$$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\vec{a} + (x - 9y)\vec{b} + (-x + 7y)\vec{c}$$

$$\Rightarrow x - y = -1, x - 9y = -5, -x + 7y = 4$$

[Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two of these three equations, we get $x = -1/2, y = 1/2$. These values also satisfy the third equation. Hence the given four points are coplanar.

9. Given, $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \Rightarrow \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = \vec{0}$

$$\Rightarrow \vec{a} \cdot (\vec{b} - \vec{c}) = 0 \Rightarrow \vec{b} - \vec{c} = \vec{0} \text{ or } \vec{a} \perp (\vec{b} - \vec{c})$$

$$\Rightarrow \vec{b} = \vec{c} \text{ or } \vec{a} \perp (\vec{b} - \vec{c})$$

...(i)

Again, given, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$

$$\Rightarrow (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = \vec{0} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$

$$\Rightarrow (\vec{b} - \vec{c}) = \vec{0} \text{ or } \vec{a} \parallel (\vec{b} - \vec{c})$$

$$\Rightarrow \vec{b} = \vec{c} \text{ or } \vec{a} \parallel (\vec{b} - \vec{c})$$

...(ii)

From (i) and (ii), we get $\vec{b} = \vec{c}$

[$\because \vec{a} \perp (\vec{b} - \vec{c})$ and $\vec{a} \parallel (\vec{b} - \vec{c})$ both cannot be simultaneously true]

10. Let l, m, n be the direction cosines of \vec{OA} . It is given that \vec{OA} is inclined at 60° to OX and at 45° to OY .

$$\therefore l = \cos 60^\circ = \frac{1}{2} \text{ and } m = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\text{But, } l^2 + m^2 + n^2 = 1 \therefore \frac{1}{4} + \frac{1}{2} + n^2 = 1 \Rightarrow n = \pm \frac{1}{2}$$

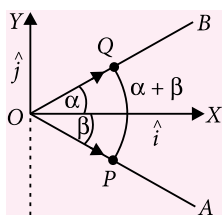
$$\text{Thus, we have } l = \frac{1}{2}, m = \frac{1}{\sqrt{2}}, n = \pm \frac{1}{2} \text{ and } |\vec{OA}| = 10$$

$$\therefore \vec{OA} = |\vec{OA}| (l\hat{i} + m\hat{j} + n\hat{k})$$

$$\Rightarrow \vec{OA} = 10 \left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \pm \frac{1}{2}\hat{k} \right) = 5\hat{i} + 5\sqrt{2}\hat{j} \pm 5\hat{k}$$

11. Let OX and OY be two mutually perpendicular lines in the plane of the paper. Let \hat{i} and \hat{j} be the unit vectors along OX and OY respectively.

Let OA and OB be two lines in the plane of the paper making angle β and α respectively with OX , OA being below OX and OB being above OX .



Then, $\angle AOB = \alpha + \beta$

Let \vec{OP} and \vec{OQ} be the unit vectors along OA and OB respectively,

$$\text{Then } \vec{OQ} = \cos \alpha \hat{i} + \sin \alpha \hat{j}$$

$$\text{and } \vec{OP} = \cos \beta \hat{i} + \sin \beta (-\hat{j}) = \cos \beta \hat{i} - \sin \beta \hat{j}$$

Let \hat{k} denote a unit vector perpendicular to the plane of the paper such that $\hat{i}, \hat{j}, \hat{k}$ form a right-handed system.

$$\text{Now } \vec{OP} \times \vec{OQ} = 1.1 \sin(\alpha + \beta) \hat{k} = \sin(\alpha + \beta) \hat{k} \quad \dots(i)$$

$$\text{Again } \vec{OP} \times \vec{OQ} = (\cos \beta \hat{i} - \sin \beta \hat{j}) \times (\cos \alpha \hat{i} + \sin \alpha \hat{j})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \beta & -\sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} = (\sin \alpha \cos \beta + \cos \alpha \sin \beta) \hat{k} \quad \dots(ii)$$

From (i) and (ii), we have

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

12. The resultant of given vectors is given by

$$(\vec{a} + \vec{b} + \vec{c}) = 3\hat{i} + 5\hat{j} + 4\hat{k}$$

$$\therefore |\vec{a} + \vec{b} + \vec{c}| = \sqrt{(3^2 + 5^2 + 4^2)} = 5\sqrt{2}$$

\therefore Unit vector in the direction of the resultant

$$= \frac{1}{5\sqrt{2}} (3\hat{i} + 5\hat{j} + 4\hat{k})$$

$$\text{Let } \vec{r} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 1 \end{vmatrix} = -4\hat{i} - 4\hat{j} + 4\hat{k}$$

$$\therefore |\vec{r}| = \sqrt{(-4)^2 + (-4)^2 + 4^2} = 4\sqrt{3}$$

Let θ be the angle between \vec{r} and \vec{c} , then

$$\theta = \cos^{-1} \left\{ \frac{\vec{r} \cdot \vec{c}}{|\vec{r}| |\vec{c}|} \right\}$$

$$= \cos^{-1} \left\{ \frac{(-4\hat{i} - 4\hat{j} + 4\hat{k}) \cdot (3\hat{i} + 5\hat{j})}{(4\sqrt{3}) \cdot (\sqrt{10})} \right\} \text{ or } \theta = \cos^{-1} \left(\frac{-4}{\sqrt{30}} \right)$$

$$\therefore \text{Angle between } \vec{r} \text{ and } \vec{c} = \pi - \cos^{-1} \left(\frac{4}{\sqrt{30}} \right)$$

13. Let \vec{b}, \vec{c} be the position vectors of B and C with respect to A as the origin of reference.

Therefore, the position vectors of D, E, F are

$$\frac{1}{2}(\vec{b} + \vec{c}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{b} \text{ respectively.}$$

Also the position vector of the point G , the centroid, is

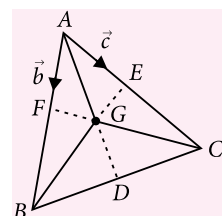
$$\frac{1}{3}(0 + \vec{b} + \vec{c}) = \frac{1}{3}(\vec{b} + \vec{c})$$

$$\text{Therefore, area of } \Delta AGB = \frac{1}{2}(\vec{AB} \times \vec{AG})$$

$$= \frac{1}{2} \left| \vec{b} \times \frac{1}{3}(\vec{b} + \vec{c}) \right| = \frac{1}{6} \left| \vec{b} \times \vec{c} \right| = \frac{1}{3} \text{ar}(\Delta ABC).$$

$$\text{Similarly, we can show that } \text{ar}(\Delta BGC) = \frac{1}{3} \text{ar}(\Delta ABC)$$

$$\text{and } \text{ar}(\Delta CGA) = \frac{1}{3} \text{ar}(\Delta ABC).$$



MPP-7 CLASS XII

ANSWER KEY

- | | | | | |
|---------|-----------|------------|--------------|---------|
| 1. (c) | 2. (b) | 3. (b) | 4. (d) | 5. (a) |
| 6. (a) | 7. (a,c) | 8. (b,c,d) | 9. (a,b,c,d) | |
| 10. (a) | 11. (a,b) | 12. (a,d) | 13. (c,d) | 14. (b) |
| 15. (c) | 16. (d) | 17. (8) | 18. (9) | 19. (3) |
| 20. (1) | | | | |

$$\begin{aligned}
 14. \text{ (i)} \quad (\vec{a} \times \vec{b})^2 &= |\vec{a} \times \vec{b}|^2 \quad (\because \vec{a}^2 = |\vec{a}|^2) \\
 &= (|\vec{a}| |\vec{b}| \sin \theta)^2 = (|\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta) \\
 &= a^2 b^2 \sin^2 \theta = a^2 b^2 (1 - \cos^2 \theta) = a^2 b^2 - a^2 b^2 \cos^2 \theta \\
 &= a^2 b^2 - (ab \cos \theta)^2 = a^2 b^2 - (|\vec{a}| |\vec{b}| \cos \theta)^2 \\
 &= a^2 b^2 - (\vec{a} \cdot \vec{b})^2
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad (\vec{a} \times \vec{b})^2 &= a^2 b^2 - (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 \\
 &= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}
 \end{aligned}$$

15. Let OABC be the parallelogram.

Take $\vec{OA} = \vec{a}$ and $\vec{OC} = \vec{c}$

Then $\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + \vec{OC} = \vec{a} + \vec{c}$

and $\vec{CA} = \vec{CO} + \vec{OA} = \vec{OA} - \vec{OC} = \vec{a} - \vec{c}$

It is given that $\vec{a} + \vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$... (i)

and $\vec{a} - \vec{c} = \hat{i} - 3\hat{j} + 4\hat{k}$... (ii)

Adding (i) and (ii), we get

$$2\vec{a} = 4\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\Rightarrow \vec{a} = 2\hat{i} - \hat{j} + \hat{k} \quad \text{... (iii)}$$

Subtracting (ii) from (i), we get

$$2\vec{c} = 2\hat{i} + 4\hat{j} - 6\hat{k} \Rightarrow \vec{c} = \hat{i} + 2\hat{j} - 3\hat{k} \quad \text{... (iv)}$$

Now, $\vec{a} \times \vec{c} = (2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} - 3\hat{k})$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \hat{i} + 7\hat{j} + 5\hat{k}$$

\therefore Area of the parallelogram OABC

$$= |\vec{a} \times \vec{c}| = |\hat{i} + 7\hat{j} + 5\hat{k}| = \sqrt{75} = 5\sqrt{3}$$

16. Consider $[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}]$

$$\begin{aligned}
 &= (\vec{a} + \vec{b}) \cdot [(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})] \\
 &= (\vec{a} + \vec{b}) \cdot [(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a})] \\
 &= (\vec{a} + \vec{b}) \cdot [\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}] \quad [\because \vec{c} \times \vec{c} = \vec{0}] \\
 &= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) + \vec{b} \cdot (\vec{b} \times \vec{c}) \\
 &\quad + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a}) \\
 &= [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{b} \vec{a}] + [\vec{a} \vec{c} \vec{a}] + [\vec{b} \vec{b} \vec{c}] + [\vec{b} \vec{b} \vec{a}] + [\vec{b} \vec{c} \vec{a}] \\
 &= [\vec{a} \vec{b} \vec{c}] + [\vec{b} \vec{c} \vec{a}] = 2 [\vec{a} \vec{b} \vec{c}] \quad \text{... (i)}
 \end{aligned}$$

Given, $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors

$$\Rightarrow [\vec{a} \vec{b} \vec{c}] \neq 0 \quad \text{... (ii)}$$

$$\Rightarrow [\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] \neq 0 \quad \text{[From (i)]}$$

$\Rightarrow \vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}$ are non-coplanar vectors.

Now, consider

$$\begin{aligned}
 [\vec{a} - \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] &= (\vec{a} - \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\} \\
 &= (\vec{a} - \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} \\
 &= (\vec{a} - \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\} \\
 &= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) \\
 &\quad - \vec{b} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{b} \times \vec{a}) - \vec{b} \cdot (\vec{c} \times \vec{a})
 \end{aligned}$$

$$\begin{aligned}
 &= [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{b} \vec{a}] + [\vec{a} \vec{c} \vec{a}] - [\vec{b} \vec{b} \vec{c}] - [\vec{b} \vec{b} \vec{a}] - [\vec{b} \vec{c} \vec{a}] \\
 &= [\vec{a} \vec{b} \vec{c}] - [\vec{b} \vec{c} \vec{a}] = [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{c}] = 0
 \end{aligned}$$

Hence $\vec{a} - \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}$, are not non-coplanar i.e., they are coplanar.

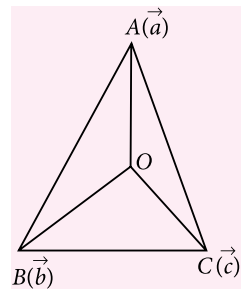
17. Let OABC be a tetrahedron. Taking O as the origin, let the position vectors of the vertices A, B, C be \vec{a}, \vec{b} and \vec{c} respectively. Let G, G_1, G_2, G_3 be the centroids of the faces ABC, OAB, OBC and OCA respectively. Then,

$$\text{Position vector of } G = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

$$\text{Position vector of } G_1 = \frac{\vec{a} + \vec{b}}{3}$$

$$\text{Position vector of } G_2 = \frac{\vec{b} + \vec{c}}{3}$$

$$\text{Position vector of } G_3 = \frac{\vec{c} + \vec{a}}{3}$$



Now, P.V. of a point dividing OG in the ratio 3 : 1

$$\begin{aligned}
 &= \frac{3 \left(\frac{\vec{a} + \vec{b} + \vec{c}}{3} \right) + 1 \cdot \vec{0}}{3 + 1} = \frac{\vec{a} + \vec{b} + \vec{c}}{4}
 \end{aligned}$$

P.V. of a point dividing AG_1 in the ratio 3 : 1

$$\begin{aligned}
 &= \frac{3 \left(\frac{\vec{a} + \vec{b}}{3} \right) + 1 \cdot \vec{a}}{3 + 1} = \frac{\vec{a} + \vec{b} + \vec{c}}{4}
 \end{aligned}$$

P.V. of a point dividing BG_2 in the ratio 3 : 1

$$\begin{aligned}
 &= \frac{3 \left(\frac{\vec{b} + \vec{c}}{3} \right) + 1 \cdot \vec{b}}{3 + 1} = \frac{\vec{a} + \vec{b} + \vec{c}}{4}
 \end{aligned}$$

P.V. of a point dividing CG_1 in the ratio 3 : 1

$$= \frac{3\left(\frac{\vec{a} + \vec{b}}{3}\right) + 1 \cdot \vec{c}}{3+1} = \frac{\vec{a} + \vec{b} + \vec{c}}{4}$$

Thus, the point having position vector $\frac{\vec{a} + \vec{b} + \vec{c}}{4}$ is common to OG , AG_2 , BG_3 and CG_1 . Hence, the line joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.

$$18. \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -\hat{i} + 7\hat{j} + 5\hat{k}$$

$$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = (-\hat{i} + 7\hat{j} + 5\hat{k}) \times (\hat{i} - 2\hat{j} + 2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 24\hat{i} + 7\hat{j} - 5\hat{k} \quad \dots(i)$$

$$\text{Again, } \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = -5\hat{j} - 5\hat{k}$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = (3\hat{i} - \hat{j} + 2\hat{k}) \times (-5\hat{j} - 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{vmatrix} = 15(\hat{i} + \hat{j} - \hat{k}) \quad \dots(ii)$$

From (i) and (ii), we conclude that

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

$$19. \text{ Given } \vec{OA} = \hat{i} + \hat{j} + \hat{k}, \vec{OB} = \hat{i} - \hat{j} + 2\hat{k}$$

$$\text{and } \vec{OC} = 0\hat{i} + 2\hat{j} - \hat{k} \text{ where } O \text{ is the origin}$$

$$\therefore \vec{AB} = \vec{OB} - \vec{OA} = 0\hat{i} - 2\hat{j} + \hat{k}$$

$$\text{and } \vec{AC} = \vec{OC} - \vec{OA} = -\hat{i} + \hat{j} - 2\hat{k}$$

Let the vector $(1, 0, 1)$ be $\vec{\alpha}$, then $\vec{\alpha} = \hat{i} + 0\hat{j} + \hat{k}$

Let $\vec{\beta} = \vec{AB} \times \vec{AC}$, then

$$\vec{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{vmatrix} = 3\hat{i} - \hat{j} - 2\hat{k}$$

Let \vec{a} be a unit vector parallel to the plane ABC and perpendicular to vector $\vec{\alpha}$.

Since $\vec{\alpha}$ is parallel to plane ABC and $\vec{\beta}$ is perpendicular to plane ABC , therefore $\vec{a} \perp \vec{\beta}$.

Thus, $\vec{a} \perp \vec{\beta}$ and $\vec{a} \perp \vec{\alpha}$ and hence $\vec{a} \parallel \vec{\beta} \times \vec{\alpha}$

$$\text{Now, } \vec{\beta} \times \vec{\alpha} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ 1 & 0 & 1 \end{vmatrix} = -\hat{i} - 5\hat{j} + \hat{k}$$

$$\text{and } |\vec{\beta} \times \vec{\alpha}| = 3\sqrt{3}$$

$$\therefore \vec{a} = \frac{\vec{\beta} \times \vec{\alpha}}{|\vec{\beta} \times \vec{\alpha}|} = \frac{1}{3\sqrt{3}}(-\hat{i} - 5\hat{j} + \hat{k})$$

20. If given points are collinear, then there exist scalars x, y, z such that $x(l_1\vec{a} + m_1\vec{b}) + y(l_2\vec{a} + m_2\vec{b})$

$$+ z(l_3\vec{a} + m_3\vec{b}) = \vec{0}, \text{ where } x + y + z = 0$$

$$\Rightarrow (l_1x + l_2y + l_3z)\vec{a} + (m_1x + m_2y + m_3z)\vec{b} = \vec{0},$$

$$\Rightarrow l_1x + l_2y + l_3z = 0, m_1x + m_2y + m_3z = 0,$$

[$\because \vec{a}, \vec{b}$ are non-collinear vectors]

Thus, we have, $x + y + z = 0$

...(i)

$$l_1x + l_2y + l_3z = 0 \quad \dots(ii)$$

$$\text{and } m_1x + m_2y + m_3z = 0 \quad \dots(iii)$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{x}{l_2m_3 - l_3m_2} = \frac{y}{l_3m_1 - l_1m_3} = \frac{z}{l_1m_2 - l_2m_1} = \lambda \text{ (say)}$$

$$\Rightarrow x = \lambda(l_2m_3 - l_3m_2), y = \lambda(l_3m_1 - l_1m_3)$$

$$z = \lambda(l_1m_2 - l_2m_1)$$

Substituting the values of x, y, z in (i), we get

$$(l_2m_3 - l_3m_2) + (l_3m_1 - l_1m_3) + (l_1m_2 - l_2m_1) = 0$$

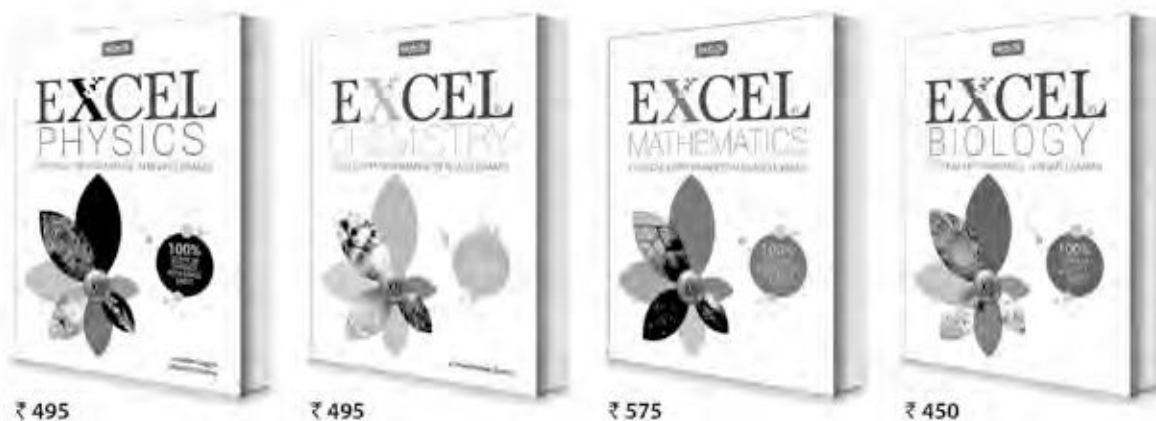
$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

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MPP-7 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Definite Integration & Application of Integrals

Total Marks : 80

Time Taken : 60 Min.

Only One Option Correct Type

1. The value of $\lim_{n \rightarrow \infty} \sum_{r=1}^{4n} \frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r} + 4\sqrt{n})^2}$ is equal to

- (a) $\frac{1}{35}$ (b) $\frac{1}{14}$
(c) $\frac{1}{10}$ (d) $\frac{1}{5}$

2. The area of the loop of the curve $ay^2 = x^2(a-x)$ is

- (a) $4a^2$ sq. units (b) $\frac{8a^2}{15}$ sq. units
(c) $\frac{16a^2}{9}$ sq. units (d) None of these

3. If $I_1 = \int_{-100}^{101} \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$

and $I_2 = \int_{-100}^{101} \frac{dx}{5+2x-2x^2}$, then $\frac{I_1}{I_2}$ is

- (a) 2 (b) $\frac{1}{2}$ (c) 1 (d) $-\frac{1}{2}$

4. Let $f(x) = \begin{cases} \min\{(x-1)^2, 2x(1-x)\}, & \text{if } 0 \leq x < \frac{1}{2} \\ \min\{2x(1-x), x^2\} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$

Then the area bounded by the curve and x -axis, is

- (a) $\frac{40}{118}$ sq. unit (b) $\frac{21}{108}$ sq. unit
(c) $\frac{40}{128}$ sq. unit (d) $\frac{31}{108}$ sq. unit

5. $\int_{-\pi/3}^0 \left[\cot^{-1}\left(\frac{2}{2\cos x - 1}\right) + \cot^{-1}\left(\cos x - \frac{1}{2}\right) \right] dx$ is equal to

- (a) $\frac{\pi^2}{6}$ (b) $\frac{\pi^2}{3}$
(c) $\frac{\pi^2}{8}$ (d) $\frac{3\pi^2}{8}$

6. The integral $\int_{-1/2}^{1/2} \left\{ [x] + \ln\left(\frac{1+x}{1-x}\right) \right\} dx$ equals

- (a) $-\frac{1}{2}$ (b) 0
(c) 1 (d) $2 \ln(1/2)$

One or More Than One Option(s) Correct Type

7. Which of the following have the same bounded area ?

- (a) $f(x) = \sin x, g(x) = \sin^2 x$, where $0 \leq x \leq 10\pi$
(b) $f(x) = \sin x, g(x) = |\sin x|$, where $0 \leq x \leq 20\pi$
(c) $f(x) = |\sin x|, g(x) = \sin^3 x$, where $0 \leq x \leq 10\pi$
(d) None of these

8. If $I_n = \int_0^{\pi/4} \tan^n x dx$, ($n > 1$) and is an integer, then

- (a) $I_n + I_{n-2} = \frac{1}{n+1}$
(b) $I_n + I_{n-2} = \frac{1}{n-1}$
(c) $I_2 + I_4, I_4 + I_6, \dots$, are in H.P.
(d) $\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

9. $C_1 : x^2 + y^2 - 2x - 4y + 1 = 0$, $C_2 : x^2 + y^2 + 8x = 0$ be two circles, then

- (a) the length of their common chord is $\sqrt{\frac{175}{29}}$
 (b) the length of their common tangents is 5
 (c) the centre of C_2 is an interior point of C_1
 (d) the area of C_2 is more than C_1

10. If $f(x) = \int_0^x (\cos(\sin t) + \cos(\cos t)) dt$, then $f(x + \pi)$ is

- (a) $f(x) + f(\pi)$ (b) $f(x) + 2f(\pi)$
 (c) $f(x) + f\left(\frac{\pi}{2}\right)$ (d) None of these

11. If $I = \int_0^{1/2} \frac{1}{\sqrt{1-x^{2n}}} dx$ for $n \geq 1$, then

- (a) $I < 1$ (b) $I > \frac{1}{2}$ (c) $I > 1$ (d) $I < \frac{1}{2}$

12. If $A_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx$, $B_n = \int_0^{\pi/2} \left(\frac{\sin nx}{\sin x}\right)^2 dx$,

for $n \in \mathbb{N}$, then

- (a) $A_{n+1} = A_n$ (b) $B_{n+1} = B_n$
 (c) $A_{n+1} - A_n = B_{n+1}$ (d) $B_{n+1} - B_n = A_{n+1}$

13. The area bounded by $y = \log_e x$, x -axis and the ordinate $x = e$ is equal to

- (a) 4 sq. unit (b) $\frac{1}{2}$ sq. unit
 (c) 1 sq. unit (d) $\int_1^e \log x dx$

Comprehension Type

Two curves $C_1 \equiv [f(y)]^{2/3} + [f(x)]^{1/3} = 0$ and $C_2 \equiv [f(y)]^{2/3} + [f(x)]^{2/3} = 12$, satisfying the relation $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$.

14. The area bounded by C_1 and C_2 is
 (a) $2\pi - \sqrt{3}$ sq. units (b) $2\pi + \sqrt{3}$ sq. units
 (c) $\pi + \sqrt{6}$ sq. units (d) $2\sqrt{3} - \pi$ sq. units
 15. The area bounded by C_1 and $x + y + 2 = 0$ is
 (a) $\frac{5}{2}$ sq. units (b) $\frac{7}{2}$ sq. units
 (c) $\frac{9}{2}$ sq. units (d) None of these

Matrix Match Type

16. Match the following.

Column I		Column II	
P.	$\int_{-1}^1 \frac{dx}{1+x^2} =$	1.	$2(\sqrt{2}-1)$
Q.	$\int_0^{\pi/2} \sqrt{1-\sin 2x} dx =$	2.	$\frac{\pi}{2}(\sqrt{2}-1)$
R.	$\int_0^{\pi/2} x\sqrt{1-\sin 2x} dx =$	3.	$\frac{\pi}{3}$
S.	$\int_1^2 \frac{dx}{x\sqrt{x^2-1}} =$	4.	$\frac{\pi}{2}$

	P	Q	R	S
(a)	2	3	1	4
(b)	2	3	4	1
(c)	1	4	3	2
(d)	4	1	2	3

Integer Answer Type

17. The area bounded by the curves $y = x(x-3)^2$ and $y = x$ is _____ (in sq. units)

18. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x}) \left(\frac{12+9x^2}{1+x^2}\right) dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ is

19. Definite integration of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx = \frac{a}{b}$, then $a+b$ is _____.

20. If $\int_1^4 (\{x\})^{[x]} dx = \frac{a}{b}$, then $a-b$ is equal to _____.
 where $\{ \cdot \}$ and $[\cdot]$ denote the fractional part and the greatest integer functions respectively.

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted
 No. of questions correct
 Marks scored in percentage

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY !	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 179

JEE MAIN

1. If a regular polygon of n sides has the circumradius R and inradius r then each side of the polygon is equal to

- (a) $2(R + r)\tan \frac{\pi}{2n}$ (b) $2r \tan \frac{\pi}{4}$
(c) $3R \tan \frac{\pi}{2n}$ (d) $(R + r)\tan \pi$

2. $\lim_{x \rightarrow \infty} \left(\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \dots + n^{1/x}}{n} \right)^{nx} =$
(a) $n!$ (b) $3n^2!$ (c) 0 (d) ∞

3. If $I = \int_0^2 \max\{\ln(1+x^2), 1\} dx$, then

$$I - 2[\ln 5 + \sqrt{e-1} - \tan^{-1} \sqrt{e-1} + \tan^{-1} 2] =$$

(a) 0 (b) 1 (c) 2 (d) -4

4. If a be the digit at unit's place in $11^{2012} + 23^{2012} - 3^{2012}$,

$$\text{then } \int_{a-1}^a \frac{dx}{\sqrt{1-x^2} - x + \frac{1}{x}} =$$

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

5. The sum of areas of all triangles whose vertices are also vertices of a cube of unit edge is $m + \sqrt{n} + \sqrt{p}$, where $m + n + p$ is

- (a) 320 (b) 332 (c) 342 (d) 348

JEE ADVANCED

6. Let $z = \cos 1^\circ + i \sin 1^\circ$, $A = \sum_{r=1}^{45} \operatorname{Re}(z^{2r-1})$

$$\text{and } B = \sum_{r=1}^{45} \operatorname{Im}(z^{2r-1})$$

- (a) $A = B$ (b) $A^2 + B^2 = 1$
(c) $\frac{1}{A^2} + \frac{1}{B^2} = 1$ (d) $\frac{1}{A} = i(\bar{z} - z)$

COMPREHENSION

A function $f: R \rightarrow R$ satisfies the following conditions:

- (i) $f(-x) = f(x)$ (ii) $f(x+2) = f(x)$

- (iii) $g(x) = \int_0^x f(t) dt$ and $g(1) = a$. Then

7. The value of $g(x+2) - g(2)$ is
(a) $3g(x)$ (b) $2g(x)$
(c) $g(x)$ (d) none of these

8. The value of $g(2)$ and $g(5)$ (respectively) in terms of a , is
(a) $2a, 5a$ (b) a, a
(c) $a/2, a/5$ (d) none of these

INTEGER TYPE

9. The image of the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$ in the plane $x + y + z = 1$ meets the x - z plane at the point (a, b, c) where c is

MATRIX MATCH

10. Match the following.

	List-I	List-II
P.	$5^2 5^4 5^6 \dots 5^{2x} = (0.04)^{-28}$	1. $3 \log_3 5$
Q.	$x^2 = (0.2)^{\log_{\sqrt{5}} \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right)}$	2. 4
R.	$x = (0.16)^{\log_{5/2} \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \right)}$	3. 2
S.	$3^{x-1} + 3^{x-2} + 3^{x-3} + \dots = 2 \left(5^2 + 5 + 1 + \frac{1}{5} + \frac{1}{5^2} + \dots \right)$	4. 7

	P	Q	R	S
(a)	2	1	4	3
(b)	4	3	2	1
(c)	2	1	3	4
(d)	2	4	3	1

See Solution Set of Maths Musing 178 on page no 84

JEEWORKCUTS

PAPER-I

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct ans. 3 marks & wrong ans., no negative mark]

- The solution of $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying $y(1) = 1$ is
 (a) a hyperbola (b) a circle
 (c) $y^2 = x(1+x) - 1$ (d) $(x-2)^2 + (y-3)^2 = 5$
- If the antiderivative of $\sin^{-1} \sqrt{\frac{x}{x+1}}$ is $x \sin^{-1} \sqrt{\frac{x}{x+1}} - \sqrt{x} + f(x) + C$ then
 (a) $f(x) = \sin^{-1} x, g(x) = \sqrt{x}$
 (b) $g(x) = \sqrt{x+1}, f(x) = \tan^{-1} x$
 (c) $f(x) = \tan^{-1} x, g(x) = \sqrt{x}$
 (d) none of these
- The value of the integral $\int_0^1 e^{x^2} dx$ is
 (a) less than e (b) greater than e
 (c) less than 1 (d) greater than 1
- The value of the integral $\int_0^{\pi/4} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ is
 (a) $\frac{1}{ab} \tan^{-1} \left(\frac{b}{a} \right) (a > 0, b > 0)$
- $\frac{1}{ab} \tan^{-1} \left(\frac{b}{a} \right) (a < 0, b < 0)$
 (c) $\frac{\pi}{4} (a=1, b=1)$ (d) $\frac{1}{ab} \tan^{-1} \left(\frac{a}{b} \right) + \frac{1}{ab}$
- The graph of the function $y = f(x)$ passing through the point $(0, 1)$ and satisfying the differential equation $\frac{dy}{dx} + y \cos x = \cos x$ is such that :
 (a) it is a constant function
 (b) it is periodic
 (c) it is neither an even nor an odd function
 (d) it is continuous and differentiable for all x .
- The differential equation of all parabolas each of which has a latus rectum ' $4a$ ' and whose axes are parallel to x -axis is
 (a) of degree 2 and order 1
 (b) of order 2 and degree 3
 (c) $2a \frac{d^2 x}{dy^2} = 1$ (d) $2a \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^3 = 0$
- $\int_0^{\pi/2} \left(\frac{1 + \sin 3x}{1 + 2 \sin x} \right) dx$
 (a) $\int_0^{\pi/2} \frac{1 - \cos 3x}{1 + 2 \cos x} dx$ (b) $\int_0^{\pi/2} \frac{\cos 3x + 1}{2 \cos x - 1} dx$
 (c) $\frac{\pi}{2}$ (d) 1
- The integral of $\frac{1}{\sin^2 x + \tan^2 x}$ must be

$$(a) -\frac{1}{2} \left[\tan x + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \tan x \right) \right] + c$$

$$(b) -\frac{1}{2} \left[\cot x + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \tan x \right) \right] + c$$

$$(c) - \left[\cot x - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \tan x \right) \right] + c$$

$$(d) -\frac{1}{2} \left[\cot x - \frac{1}{\sqrt{2}} \cot^{-1} \left(\frac{1}{\sqrt{2}} \tan x \right) \right] + c$$

9. Orthogonal trajectories of family of parabolas $y^2 = 4a(x + a)$ where 'a' is an arbitrary constant is :

$$(a) ax^2 = 4cy \quad (b) x^2 + y^2 = a^2$$

$$(c) y = ce^{-x/2a} \quad (d) axy = c^2$$

where c is a constant.

10. $\int \left(\frac{\ln x - 1}{(\ln x)^2 + 1} \right)^2 dx$ is equal to

$$(a) \frac{x}{x^2 + 1} + c \quad (b) \frac{\ln x}{(\ln x)^2 + 1}$$

$$(c) \frac{x}{(\ln x)^2 + 1} + c \quad (d) e^x \left(\frac{x}{x^2 + 1} \right) + c$$

INTEGER ANSWER TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct ans. 3 marks & wrong ans., no negative mark]

11. If $f(x) = \max \{2 - x, 2, 1 + x\}$, then $\int_{-1}^2 f(x) dx =$

12. The area bounded by the curves $y = \ln x$, x -axis and $x = e$ is

13. If $g(x) = \frac{1}{x} \int_2^x \{3t - 2g'(t)\} dt$, then $2g'(2) =$

14. Find the area enclosed by the curve $[x] + [y] = 4$ in the 1st quadrant (where $[.]$ denotes greatest integer function).

15. The order of the differential equation whose general solution is given by $y = C_1 + C_2 \cos x + C_3 + C_4 e^{x+C_5}$ where C_1, C_2, C_3, C_4, C_5 are arbitrary constants is

16. The value of integral, $\int_0^{\pi/2} \sin 2x \tan^{-1} \sin x dx$ must be equal to $\frac{\pi}{2} - k$, where k is

17. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is

18. The value of $\int_{-2}^2 |1 - x^2| dx$ is

19. If $\int_0^{\pi/2} f(\sin 2x) \sin x dx = A\sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x dx$ then the value of A is

20. The value of $y(\sqrt{8}) - \frac{1}{9}$ if $(1 + x^2) \frac{dy}{dx} = x(1 - y)$, $y(0) = 4/3$ is

PAPER-II

ONLY ONE OPTION CORRECT TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. [Correct ans. 3 marks and wrong ans. -1]

1. The area bounded by the curves $f(x) = \sin^{-1}(\sin x)$ and $g(x) = [\sin^{-1}(\sin x)]$ in the interval $[0, \pi]$, where $[.]$ is a greatest integer function, is
(a) $\pi/2$ (b) $(\pi/2 - 1)^2$
(c) π (d) $(\pi/4 - 1)^2$

2. The value of $\int_0^2 [x^2 - 1] dx$, where $[x]$ denotes the greatest integer function, is given by
(a) $3 - \sqrt{3} - \sqrt{2}$ (b) $2 - \sqrt{3}$
(c) $4 - \sqrt{3} - \sqrt{2}$ (d) none of these

3. $\int_0^{\infty} \frac{x \log x}{(1 + x^2)^2} dx$ is equal to

(a) 1 (b) 0 (c) 2 (d) none of these

4. If $A = \int_0^{3\pi/2} \frac{\cos x}{\cos x - \sin x} dx$, $B = \int_0^{3\pi/2} \frac{\sin x}{\cos x - \sin x} dx$

then the value of $A + B$ is

(a) $3\pi/2$ (b) $\pi/4$ (c) 0 (d) π

5. The antiderivative of $f(x) = \frac{1}{3 + 5 \sin x + 3 \cos x}$, whose graph passes through the point $(0, 0)$ is

(a) $\frac{1}{5} \ln \left| 1 - \frac{5}{3} \tan \frac{x}{2} \right|$ (b) $\frac{1}{5} \ln \left| 1 + \frac{5}{3} \tan \frac{x}{2} \right|$

(c) $\frac{1}{5} \ln \left| 1 + \frac{5}{3} \cot \frac{x}{2} \right|$ (d) none of these

6. $\int \frac{xe^x}{(1+x)^2} dx =$

(a) $\frac{e^x}{x+1} + c$ (b) $e^x(x+1) + c$
(c) $-\frac{e^x}{(x+1)^2} + c$ (d) $\frac{e^x}{1+x^2} + c$

7. A solution of $y = 2x \left(\frac{dy}{dx} \right) + x^2 \left(\frac{dy}{dx} \right)^4$ is

(a) $y = 2c^{1/2} x^{1/4} + c$ (b) $y = 2\sqrt{c} x^2 + c^2$
(c) $y = 2\sqrt{c}(x+1)$ (d) $y = 2\sqrt{cx} + c^2$

8. If $\int \tan^7 x dx = f(x) + \log |\cos x|$, then

(a) $f(x)$ is a polynomial of degree 8 in $\tan x$
(b) $f(x)$ is a polynomial of degree 5 in $\tan x$
(c) $f(x)$ is a polynomial of degree 6 in $\tan x$

(d) $f(x) = \frac{\tan^6 x}{6} - \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + \log |\cos x| + C$

9. If $I_{m,n} = \int \cos^m x \sin nx dx$, then $7 I_{4,3} - 4 I_{3,2} =$

(a) constant (b) $-\cos^2 x + C$
(c) $-\cos^4 x \cos 3x + C$ (d) $\cos 7x - \cos 4x + C$

10. If $f(x)$ is an even and differentiable function, then

the value of $\int_{-1}^1 x^3 f(x) + x f''(x) + 3 dx =$

(a) 6 (b) 2 (c) 0 (d) 1

COMPREHENSION TYPE

This section contains 2 paragraph. Based upon each of the paragraphs 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [Correct ans. 3 marks & wrong ans. -1]

Paragraph for Q. No. 11 and 12

Consider the integral $\int \frac{\phi(x) dx}{\sqrt{ax^2 + bx + c}}$ where $\phi(x)$ is a polynomial in x .

If $\phi(x)$ is a polynomial of degree n , then there exists a polynomial $f(x)$ of degree $(n-1)$ and a constant such that

$$\int \frac{\phi(x) dx}{\sqrt{ax^2 + bx + c}} = f(x) \sqrt{ax^2 + bx + c} + D \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

11. Differentiating both sides with respect to x and multiply by $\sqrt{ax^2 + bx + c}$, we get

(a) $\phi(x) = f'(x)(ax^2 + bx + c) + \frac{1}{2}(2ax + b)f(x) + D$

(b) $\phi(x) = f'(x)(ax^2 + bx + c) - \frac{1}{2}(2ax + b)f(x) + D$

(c) $\phi(x) = \frac{1}{2} f'(x)(ax^2 + bx + c) + \frac{1}{2}(2ax + b)f(x) + D$

(d) none of these

12. Now apply this method to evaluate the given integral.

If $\int \frac{(x^3 + 4x^2 - 6x + 3) dx}{\sqrt{5 + 6x - x^2}}$
 $= (Ax^2 + Bx + C) \sqrt{(5 + 6x - x^2)} + D \sin^{-1} \left(\frac{3-x}{\sqrt{14}} \right)$

then

(a) $A = \frac{2}{3}$ (b) $B = \frac{9}{5}$ (c) $C = \frac{1}{6}$ (d) $C = -\frac{227}{6}$

Paragraph for Q. No. 13 and 14

An even function f is defined and integrable everywhere and is periodic with period 2.

Also, function $g(x) = \int_0^x f(t) dt$ and $g(1) = A$

13. Function $g(x)$ is

(a) odd (b) even
(c) neither even nor odd
(d) can't be determined

14. Value of $g(2)$ in terms of A is

(a) $2A$ (b) $A/2$ (c) $4A$ (d) $A/4$

Paragraph for Q. No. 15 and 16

Let $f(x) = \frac{2}{x^3 + 6x^2 + 11x + 6}$.

15. Then $\int_0^1 f(x) dx$ is equal to

(a) $3 \ln 3 - 5 \ln 2$ (b) $5 \ln 2 - 3 \ln 3$
(c) $5 \ln 2 + 3 \ln 3$ (d) none of these

16. $f(n)$ is equal to

(a) $\frac{{}^nC_0}{3} - \frac{{}^nC_1}{4} + \frac{{}^nC_2}{5} - \dots$ to $(n+1)$ terms

(b) $\frac{{}^nC_1}{3} - \frac{{}^nC_2}{4} + \frac{{}^nC_3}{5} - \dots$ to $(n+1)$ terms

(c) $\frac{{}^nC_3}{3} - \frac{{}^nC_4}{4} + \frac{{}^nC_5}{5} - \dots$ to $(n+1)$ terms

(d) none of these

MATRIX MATCH TYPE

This section contains 4 questions, each having two matching columns. Choices for the correct combination of elements from column-I and column-II are given as options (a), (b), (c) and (d), out of which one is correct. [Correct ans. 3 marks & wrong ans. -1]

17. Match the following.

Column I		Column II	
(P)	The equation of curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$. The area of a loop of the above curve is	1.	$4a^2$
(Q)	The area of the curve $a^2y^2 = x^2(a^2 - x^2)$ is	2.	$2a^2\left(1 + \frac{\pi}{4}\right)$
(R)	Area contained between the curve $y^2(a - x) = x^2(a + x)$ and its asymptotes is	3.	$a^2\left(\frac{\pi}{2} - 1\right)$
(S)	The area enclosed by the parabola $ay = 3(a^2 - x^2)$ and the x -axis is	4.	$\frac{4}{3}a^2$

Codes :

P	Q	R	S	P	Q	R	S
(a) 1	2	3	4	(b) 3	4	2	1
(c) 3	1	2	4	(d) 3	4	1	2

18. Match the following.

Column I		Column II	
(P)	$\int \frac{1}{\sin x - \cos x} dx$	1.	$\tan^{-1}(\tan^2 x) + C$
(Q)	$\int \frac{\sin(2x)}{\sin^4 x + \cos^4 x} dx$	2.	$\left(\frac{\ln x - \ln(1+x) + \frac{1}{1+x} - \frac{\ln x}{(1+x)^2} \right) + C$
(R)	$\int \frac{e^{\ln(1+1/x^2)}}{x^2 + \frac{1}{x^2}} dx$	3.	$\frac{1}{\sqrt{2}} \ln \tan\left(\frac{x}{2} - \frac{\pi}{8}\right) + C$
(S)	$\int \frac{\ln x}{(1+x)^3} dx$	4.	$\frac{1}{\sqrt{2}} \tan^{-1} \frac{x^2 - 1}{x\sqrt{2}} + C$

Codes :

P	Q	R	S	P	Q	R	S
(a) 3	4	1	2	(b) 3	1	2	4
(c) 3	1	4	2	(d) 4	1	2	3

19. Match the following.

Column I		Column II	
(P)	$\int \frac{e^x}{x+2} [(1+(x+2)) \ln(x+2)] dx$	1.	$\sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$

(Q)	$\int \sin^2 x \cos^3 x dx$	2.	$\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2} \ln(x^2 + 1) + c$
(R)	$\int \frac{dx}{\sqrt{2-3x-x^2}}$	3.	$e^x \ln(x+2) + c$
(S)	$\int \frac{x^5}{x^2+1} dx$	4.	$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

Codes :

P	Q	R	S	P	Q	R	S
(a) 3	4	1	2	(b) 4	1	2	3
(c) 1	2	3	4	(d) 3	1	2	4

20. Match the following.

Column I		Column II	
(P)	$\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$	1.	$2\left(1 - \frac{1}{e}\right)$
(Q)	$\int_0^{41\pi/4} \cos x dx$	2.	$\ln 4 - \ln 3$
(R)	$\int_0^1 \frac{\ln(1-x)}{x} dx$	3.	$20 + \frac{1}{\sqrt{2}}$
(S)	$\int_{1/e}^e \ln x dx$	4.	$-\frac{\pi^2}{6}$

Codes :

P	Q	R	S	P	Q	R	S
(a) 2	3	4	1	(b) 1	2	3	4
(c) 3	4	1	2	(d) 2	1	3	4

ANSWERS KEY

Paper-I

1.	(a, c)	2.	(c)	3.	(a, d)	4.	(a, b, c)
5.	(a, b, d)	6.	(c, d)	7.	(a, b, d)	8.	(b, d)
9.	(c)	10.	(c)	11.	(7)	12.	(1)
13.	(3)	14.	(5)	15.	(3)	16.	(1)
17.	(2)	18.	(4)	19.	(1)	20.	(1)

Paper-II

1.	(b)	2.	(a)	3.	(b)	4.	(c)
5.	(b)	6.	(a)	7.	(d)	8.	(d)
9.	(c)	10.	(a)	11.	(a)	12.	(d)
13.	(a)	14.	(a)	15.	(b)	16.	(a)
17.	(b)	18.	(c)	19.	(a)	20.	(a)

For detailed solution to the Sample Paper, visit our website : www.vidyalankar.org





JEE Main 2018

MOCK TEST PAPER

Series-5

Time: 1 hr 15 min.

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

Unit No.5	Topic	Syllabus In Detail
	Differential calculus	Functions, Limits, continuity & differentiability.
	Statistics & Probability	Measures of Dispersion: Calculation of mean, median, mode of grouped and ungrouped data, calculation of standard deviation, variance and mean deviation for grouped and ungrouped data. Probability: Probability of an event, addition and multiplication theorems of probability.
	Co-ordinate geometry-3D	Coordinate axes and coordinate planes in three dimension. Coordinate of a point, Distance between two points and section formula.

- The domain of $\sin^{-1}[2x^2 - 5]$ where $[\cdot]$ represents greatest integer function, is
 (a) $\left[-\sqrt{\frac{7}{2}}, -\sqrt{2}\right]$ (b) $\left[\sqrt{2}, \frac{7}{2}\right]$
 (c) $\left[-\sqrt{\frac{7}{2}}, -\sqrt{2}\right] \cup \left[\sqrt{2}, \sqrt{\frac{7}{2}}\right]$
 (d) none of these
- The range of $f(x) = {}^{(15-x)}C_{(2x-1)} + {}^{(20-3x)}C_{(4x-5)}$ is
 (a) [1300, 1400] (b) {650, 1122}
 (c) {0, 4224} (d) none of these
- If $f: [-15, 15] \rightarrow \mathbb{R}$ defined by $f(x) = \left[\frac{x^2}{a}\right] \sin x + \cos x$ (where $[\cdot]$ denotes G.I.F.) is an even function then set of values of a is given by
 (a) (225, ∞) (b) \mathbb{R}
 (c) (0, 225) (d) \emptyset
- If $f(x) = \cos(\log x)$ then the value of $f(x)f(y) - \frac{1}{2} \left[f\left(\frac{x}{y}\right) + f(xy) \right] =$
 (a) x^2 (b) $x^2 + 2x + 1$
 (c) 0 (d) $x + 1$
- Assume f is differentiable function satisfying $f(x+y) = f(x) + f(y) + xy$ and $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 3$ then $f(1) =$
 (a) 5/2 (b) 7/2 (c) -5/2 (d) -7/2
- $\lim_{x \rightarrow 0} \frac{xy\sqrt{y^2 - (y-x)^2}}{(\sqrt{8xy - 4x^2} + \sqrt{8xy})^3} =$
 (a) $\frac{1}{512}$ (b) $\frac{1}{128}$
 (c) $\frac{1}{64}$ (d) none of these
- $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4} =$
 (a) 1/3 (b) 1/6
 (c) 1/2 (d) none of these
- $\lim_{x \rightarrow 1} 2[2+x] =$ ($[\cdot]$ represents G.I.F.)
 (a) 6 (b) 4
 (c) 2 (d) does not exist

By : Sankar Ghosh, S.G.M.C, Mob : 09831244397.

9. $\lim_{x \rightarrow 0} \frac{\tan[e^2]x^4 - \tan[-e^2]x^4}{\sin^4 x} = [\text{ where } [\cdot] \text{ is G.I.F.}]$

- (a) 0 (b) 15 (c) 8 (d) 7

10. $\lim_{x \rightarrow 0} \left\{ 1 + x + \frac{f(x)}{x} \right\}^{\frac{1}{x}} = e^3$ where $f(x)$ is a polynomial

in x , then $\lim_{x \rightarrow 0} \frac{5x^2 - f(x)}{x^2} =$

- (a) 5 (b) -1 (c) 4 (d) 3

11. Let $f(x) = \begin{cases} \left[\tan\left(\frac{\pi}{4} + x\right) \right]^{1/x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ then the value

of k such that $f(x)$ hold continuity at $x = 0$

- (a) e (b) e^{-2}
(c) e^2 (d) none of these

12. If $f(x) = \begin{cases} \frac{a|x^2 - x - 2|}{2 + x - x^2}, & x < 2 \\ b, & x = 2 \\ \frac{x - [x]}{x - 2}, & x > 2 \end{cases}$ is continuous at

$x = 2$, then

- (a) $a = 1, b = 2$ (b) $a = 1, b = 1$
(c) $a = 2, b = 1$ (d) $a = b = 2$

13. Let $f(x) = [x]x$ for $-1 \leq x \leq 2$ then $f(x)$ is

- (a) discontinuous at $x = 0$
(b) differentiable at $x = 1$
(c) not differentiable at $x = 2$
(d) differentiable at $x = 2$

14. $f(x) = \frac{|x+2|}{\tan^{-1}(x+2)}$ then which is not true?

- (a) $\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^-} f(x)$
(b) $\lim_{x \rightarrow -2^+} f(x) = 1$
(c) f is continuous except at $x = -2$
(d) f is non differentiable at $x = -2$

15. $f(x) = \begin{cases} \frac{216^x - 27^x - 8^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}}, & \forall x \neq 0 \\ \lambda, & \text{at } x = 0 \end{cases}$

is continuous at $x = 0$. If $\lambda = \sqrt{\alpha} \log 2 \cdot \log 3$, then the value of α is

- (a) 1547 (b) 1152
(c) 2352 (d) 2592

16. $f(x) = \begin{cases} \frac{\sin\{\cos x\}}{x - \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$ where $\{\cdot\}$ represents the

fractional part function, then $f(x)$ is

- (a) continuous at $x = \pi/2$
(b) $\lim_{x \rightarrow \pi/2} f(x)$ exists, but f is not continuous at $x = \frac{\pi}{2}$
(c) $\lim_{x \rightarrow \pi/2} f(x)$ does not exist
(d) $\lim_{x \rightarrow \pi/2^+} f(x) = -1$

17. If \bar{x}_1 and \bar{x}_2 are means of two distribution such that $\bar{x}_1 < \bar{x}_2$ and \bar{x} is the mean of the joint distribution then

- (a) $\bar{x} = \frac{\bar{x}_1 + \bar{x}_2}{2}$ (b) $\bar{x} > \bar{x}_2$
(c) $\bar{x}_2 < \bar{x}_1$ (d) $\bar{x}_1 < \bar{x} < \bar{x}_2$

18. If S.D. of a variate x is σ then the S.D. of $\frac{ax+b}{p}$ ($\forall a, b, p \in \mathbb{R}$) is

- (a) $\frac{a}{p}\sigma_x$ (b) $\left|\frac{a}{p}\right|\sigma_x$ (c) $\left|\frac{p}{a}\right|\sigma_x$ (d) $\frac{p}{a}\sigma_x$

19. If a variate assumes the values 0, 1, 2, ..., n with frequencies ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ then mean square deviation about the value $x = 0$ is

- (a) $\frac{n(n-1)}{2}$ (b) $\frac{n^2(n-1)}{4}$
(c) $\frac{n(n+1)}{4}$ (d) $\frac{n(n+1)}{2}$

20. Let r be the range of n ($\forall n \geq 1$) observations

x_1, x_2, \dots, x_n , if $S = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}$ then

- (a) $S < r\sqrt{\frac{n^2+1}{n-1}}$ (b) $S \geq r\sqrt{\frac{n}{n-1}}$
(c) $S = r\sqrt{\frac{n}{n-1}}$ (d) $S < r\sqrt{\frac{n}{n-1}}$

21. From a set of 40 cards numbered 1 to 40, 5 cards drawn at random and arranged in ascending order of magnitude $x_1 < x_2 < x_3 < x_4 < x_5$. The probability that $x_3 = 24$ is

- (a) $\frac{{}^{16}C_2}{{}^{40}C_5}$ (b) $\frac{{}^{23}C_2}{{}^{40}C_5}$
 (c) $\frac{{}^{16}C_2 \times {}^{23}C_2}{{}^{40}C_5}$ (d) none of these

22. If $a \in [-6, 12]$, then the probability that graph of $y = x^2 + 2(a+4)x + (3a+40)$ is strictly below x -axis is

- (a) $2/3$ (b) $1/3$
 (c) $1/2$ (d) none of these

23. A and B are two events such that $P(A \cup B) = \frac{5}{6}$ and $P(A \cap B) = \frac{1}{3}$, if $P(B^c) = \frac{1}{2}$, then events are

- (a) independent (b) mutually exclusive
 (c) disjoint (d) none of these

24. The probability that Ram will alive 30 years hence is $\frac{7}{11}$ and Shyam will be alive is $\frac{7}{10}$. What is the probability that both Ram and Shyam will be dead 30 years hence?

- (a) $\frac{1}{11}$ (b) $\frac{12}{110}$
 (c) $\frac{2}{121}$ (d) none of these

25. In a group of 20 males and 5 females, 10 males and 3 females are service holders. The probability that a person selected at random from the group, is a service holder, given that the selected person is a male, is

- (a) $1/2$ (b) $2/3$ (c) $2/5$ (d) $3/5$

26. The points $(4, -5, 1)$, $(3, -4, 0)$, $(6, -7, 3)$, $(7, -8, 4)$ are vertices of a

- (a) square (b) parallelogram
 (c) rectangle (d) rhombus

27. The plane $ax + by + cz - 3 = 0$ meets the co-ordinate axes in A, B, C. The centroid of the triangle is

- (a) $(3a, 3b, 3c)$ (b) $\left(\frac{3}{a}, \frac{3}{b}, \frac{3}{c}\right)$
 (c) $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ (d) $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$

28. Ratio in which the xy -plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ is

- (a) $3 : 1$ internally (b) $3 : 1$ externally
 (c) $1 : 2$ internally (d) $2 : 1$ externally

29. If $P(3, 2, -4)$, $Q(5, 4, -6)$, and $R(9, 8, -10)$ are collinear, then R divides PQ in the ratio

- (a) $3 : 2$ internally (b) $3 : 2$ externally
 (c) $2 : 1$ internally (d) $2 : 1$ externally

30. A $(3, 2, 0)$, B $(5, 3, 2)$ and C $(-9, 6, -3)$ are the vertices of a triangle ABC. If the bisector of $\angle BAC$ meets BC at D then co-ordinates of D are

- (a) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (b) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$
 (c) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$ (d) none of these

SOLUTIONS

1. (c) : $-\frac{\pi}{2} \leq \sin^{-1}[2x^2 - 5] \leq \frac{\pi}{2} \therefore -1 \leq [2x^2 - 5] \leq 1$

$$\Rightarrow -1 \leq 2x^2 - 5 < 2 \Rightarrow 4 \leq 2x^2 < 7$$

$$\Rightarrow 2 \leq x^2 < \frac{7}{2} \Rightarrow \sqrt{2} \leq |x| \leq \sqrt{\frac{7}{2}}$$

$$\therefore |x| \geq \sqrt{2} \text{ and } |x| \leq \sqrt{\frac{7}{2}}$$

$$\Rightarrow x \leq -\sqrt{2} \cup x \geq \sqrt{2} \text{ and } -\sqrt{\frac{7}{2}} \leq x \leq \sqrt{\frac{7}{2}}$$

2. (b) : Here $f(x) = {}^{15-x}C_{2x-1} + {}^{20-3x}C_{4x-5} \dots (i)$

The function will be defined if $15 - x > 0$ and $2x - 1 \geq 0$

$$\therefore x < 15 \text{ and } x \geq \frac{1}{2}$$

$$\text{Also, } 15 - x \geq 2x - 1 \Rightarrow x \leq \frac{16}{3} = 5\frac{1}{3}$$

$$\text{And } 20 - 3x > 0 \Rightarrow x < \frac{20}{3} = 6\frac{2}{3}; 4x - 5 \geq 0 \Rightarrow x \geq \frac{5}{4}$$

$$\text{Also, } 20 - 3x \geq 4x - 5 \Rightarrow x \leq \frac{25}{7} = 3\frac{4}{7}$$

Here, $15 - x$ is an integer. So, x must be an integer.

$$\therefore \frac{5}{4} \leq x \leq \frac{25}{7} \text{ and } x \in \mathbb{N} \text{ or } \mathbb{I}^+$$

$$\Rightarrow 1\frac{1}{4} \leq x \leq 3\frac{4}{7} \Rightarrow x = 2, 3$$

Now from (i), we get

$$f(2) = {}^{13}C_3 + {}^{14}C_3 = 650 \text{ and}$$

$$f(3) = {}^{12}C_5 + {}^{11}C_7 = 1122$$

Thus the range of $f(x)$ is $\{650, 1122\}$.

3. (a) : Here $f(x) = \left[\frac{x^2}{a}\right] \sin x + \cos x$ will be even

$$\text{function if } \left[\frac{x^2}{a}\right] = 0 \Rightarrow 0 \leq \frac{x^2}{a} < 1$$

$\therefore x \in [-15, 15] \quad \therefore 0 \leq x^2 \leq 225 \Rightarrow 0 \leq \frac{x^2}{a} \leq \frac{225}{a}$
Clearly $a > 225$.

$$\begin{aligned} 4. \text{ (c) : We have, } & f(x) \cdot f(y) - \frac{1}{2} \left[f\left(\frac{x}{y}\right) + f(xy) \right] \\ &= \cos(\log x) \cos(\log y) - \frac{1}{2} \left[\cos\left(\log\left(\frac{x}{y}\right)\right) + \cos(\log(xy)) \right] \\ &= \cos(\log x) \cos(\log y) - \frac{1}{2} (\cos(\log x - \log y) + \cos(\log x + \log y)) \\ &= \cos(\log x) \cos(\log y) - \frac{1}{2} (2 \cos(\log x) \cos(\log y)) \\ &= \cos(\log x) \cos(\log y) - \cos(\log x) \cos(\log y) = 0 \end{aligned}$$

5. (b) : Given that f is differentiable

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + xh - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(h) + xh}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h)}{h} + x \Rightarrow f'(x) = 3 + x \Rightarrow \frac{df(x)}{dx} = 3 + x \\ &\Rightarrow \int df(x) = \int 3 dx + \int x dx \Rightarrow f(x) = 3x + \frac{x^2}{2} + c \\ &\Rightarrow c = 0 \text{ (As } f(0) = 0) \\ \therefore f(x) &= 3x + \frac{x^2}{2} \Rightarrow f(1) = \frac{7}{2} \end{aligned}$$

$$\begin{aligned} 6. \text{ (b) : } \lim_{x \rightarrow 0} \frac{xy\sqrt{2xy-x^2}}{x^2(\sqrt{8y-4x}+\sqrt{8y})^3} \\ &= \lim_{x \rightarrow 0} \frac{y\sqrt{2y-x}}{(\sqrt{8y-4x}+\sqrt{8y})^3} = \frac{\sqrt{2}(y)^{3/2}}{(2\sqrt{8y})^3} = \frac{\sqrt{2}}{8 \times (8)^{3/2}} = \frac{1}{128} \end{aligned}$$

7. (b)

$$8. \text{ (d) : } \lim_{x \rightarrow 1} 2[2+x] = \lim_{x \rightarrow 1} (4+2[x])$$

$[\because [x+n] = n + [x], \text{ when } n \text{ is an integer}]$

$$\text{Now, L.H.L} = \lim_{x \rightarrow 1^-} (4+2[x]) = 4+0=4$$

$$\text{and R.H.L} = \lim_{x \rightarrow 1^+} (4+2[x]) = 4+2=6$$

$\therefore \text{L.H.L} \neq \text{R.H.L} \therefore \text{Limit does not exist.}$

$$\begin{aligned} 9. \text{ (b) : } \lim_{x \rightarrow 0} \frac{\tan[e^2]x^4 - \tan[-e^2]x^4}{\sin^4 x} \\ &= \lim_{x \rightarrow 0} \frac{\tan 7x^4 - \tan(-8)x^4}{\sin^4 x} = \lim_{x \rightarrow 0} \frac{\tan 7x^4 + \tan 8x^4}{\sin^4 x} \end{aligned}$$

$$= \frac{7 \lim_{x \rightarrow 0} \frac{\tan 7x^4}{7x^4} + 8 \lim_{x \rightarrow 0} \frac{\tan 8x^4}{8x^4}}{\lim_{x \rightarrow 0} \frac{\sin^4 x}{x^4}} = \frac{7+8}{1} = 15$$

$$\begin{aligned} 10. \text{ (d) : Given that } \lim_{x \rightarrow 0} \left(1+x+\frac{f(x)}{x}\right)^{1/x} &= e^3 \\ \Rightarrow \lim_{x \rightarrow 0} \frac{1}{e^{x \rightarrow 0} x} \left(x+\frac{f(x)}{x}\right) &= e^3 \therefore \lim_{x \rightarrow 0} \frac{1}{x} \left(x+\frac{f(x)}{x}\right) = 3 \\ \Rightarrow \lim_{x \rightarrow 0} \left(1+\frac{f(x)}{x^2}\right) &= 3 \Rightarrow \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2 \end{aligned}$$

$$\text{Now, } \lim_{x \rightarrow 0} \frac{5x^2 - f(x)}{x^2} = 5 - \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5 - 2 = 3$$

11. (c) : Given that $f(x)$ is a continuous function,

$$\text{therefore } \lim_{x \rightarrow 0} \left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x} = k$$

$$\text{Let } A = \left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x} \Rightarrow \log A = \frac{1}{x} \log \tan\left(\frac{\pi}{4}+x\right)$$

$$\Rightarrow \lim_{x \rightarrow 0} (\log A) = \lim_{x \rightarrow 0} \frac{1}{x} \log \tan\left(\frac{\pi}{4}+x\right)$$

$$\Rightarrow \log\left(\lim_{x \rightarrow 0} A\right) = \lim_{x \rightarrow 0} \frac{1}{x} [\log(1+\tan x) - \log(1-\tan x)] = 2$$

$$\therefore \lim_{x \rightarrow 0} A = e^2 \Rightarrow \lim_{x \rightarrow 0} \left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x} = e^2$$

By the problem, $k = e^2$

12. (b) : The function can be redefined as

$$f(x) = \begin{cases} a, & \text{if } x < 2 \\ b, & \text{if } x = 2 \\ 1, & \text{if } x > 2 \end{cases}$$

[since in the neighbourhood of 2 on left side

$$|x^2 - x - 2| = -(x^2 - x - 2) \text{ and } x - [x] = x - 2 \text{ if } x \in (2, 3)]$$

$$\therefore f(x) \text{ is continuous at } x = 2 \Rightarrow a = b = 1$$

$$13. \text{ (c) : } f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ x, & 1 \leq x < 2 \\ 2x, & 2 \leq x < 3 \end{cases}$$

$$\text{Now, } \lim_{x \rightarrow 0^-} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = 0, f(0) = 0$$

Therefore $f(x)$ is continuous at $x = 0$

$$\text{Again } \lim_{x \rightarrow 1^-} f(x) = 0, \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} x = 1$$

$\therefore f(x)$ is discontinuous at $x = 1$ and so non differentiable at $x = 1$

Again $\lim_{x \rightarrow 2^-} f(x) = 2$, $\lim_{x \rightarrow 2^+} f(x) = 4 = f(2)$

$\therefore f$ is discontinuous at $x = 2$, so non-differentiable at $x = 2$.

14. (a) : The given function is $f(x) = \frac{|x+2|}{\tan^{-1}(x+2)}$

$$\therefore \lim_{x \rightarrow -2^+} \frac{|x+2|}{\tan^{-1}(x+2)} = \lim_{x \rightarrow -2^+} \frac{x+2}{\tan^{-1}(x+2)} = 1$$

$$\text{and } \lim_{x \rightarrow -2^-} \frac{|x+2|}{\tan^{-1}(x+2)} = - \lim_{x \rightarrow -2^-} \frac{x+2}{\tan^{-1}(x+2)} = -1$$

$$\text{Therefore } \lim_{x \rightarrow -2^+} \frac{|x+2|}{\tan^{-1}(x+2)} \neq \lim_{x \rightarrow -2^-} \frac{|x+2|}{\tan^{-1}(x+2)}$$

15. (d) : As $f(x)$ is continuous at $x = 0$.

$$\therefore f(0) = \text{RHL} = \lambda$$

$$\therefore \lambda = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{216^h - 27^h - 8^h + 1}{\sqrt{2} - \sqrt{1 + \cos h}}$$

$$= \lim_{h \rightarrow 0} \frac{(8^h - 1)(27^h - 1)(\sqrt{2} + \sqrt{1 + \cos h})}{(\sqrt{2} - \sqrt{1 + \cos h})(\sqrt{2} + \sqrt{1 + \cos h})}$$

$$= \frac{\log 8 \cdot \log 27 \times 2\sqrt{2}}{\lim_{h \rightarrow 0} \frac{2 \sin^2 h/2}{h^2}} = \frac{9(\log 2 \log 3) \times 2\sqrt{2}}{\frac{1}{2}}$$

$$= 36\sqrt{2} \log 2 \cdot \log 3 = \sqrt{\alpha} \log 2 \cdot \log 3 \Rightarrow \alpha = 2592$$

16. (c) : We have, $f(x) = \begin{cases} \frac{\sin\{\cos x\}}{x - \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$

$$\text{Now, } \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{h \rightarrow 0} \left(\frac{\sin(\sin h)}{-h} \right) = -1$$

$$\text{And } \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = \lim_{h \rightarrow 0} \frac{\sin(1 - \sin h)}{h} \neq -1$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} f(x) \text{ does not exist.}$$

17. (d) : Let \bar{x}_1 and \bar{x}_2 are the arithmetic means of two distributions respectively with n_1 and n_2 observations.

$$\text{Now the combined mean } \bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

$$\therefore \bar{x} - \bar{x}_1 = \frac{n_2}{n_2 + n_1} (\bar{x}_2 - \bar{x}_1)$$

$$\Rightarrow \bar{x} - \bar{x}_1 > 0 \text{ as } \bar{x}_2 > \bar{x}_1 \quad \dots(i)$$

$$\text{Similarly, } \bar{x} - \bar{x}_2 = \frac{n_1}{n_2 + n_1} (\bar{x}_1 - \bar{x}_2)$$

$$\Rightarrow \bar{x} - \bar{x}_2 < 0 \text{ as } \bar{x}_2 > \bar{x}_1 \quad \dots(ii)$$

$$(i) \text{ and } (ii) \Rightarrow \bar{x}_1 < \bar{x} < \bar{x}_2$$

18. (b) : Let $y = \frac{ax+b}{p} \therefore \bar{y} = \frac{a\bar{x}+b}{p}$

$$\text{Now, } y - \bar{y} = \frac{1}{p} a(x - \bar{x}) \Rightarrow (y - \bar{y})^2 = \frac{a^2}{p^2} (x - \bar{x})^2$$

$$\Rightarrow \frac{1}{n} \sum (y - \bar{y})^2 = \frac{a^2}{p^2} \frac{1}{n} \sum (x - \bar{x})^2$$

$$\therefore \text{S.D. of } y = \left| \frac{a}{p} \right| \text{S.D. of } x = \left| \frac{a}{p} \right| \sigma_x$$

19. (c) : Mean of the distribution is

$$\bar{x} = \frac{0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \dots + n \cdot {}^nC_n}{{}^nC_0 + {}^nC_1 + \dots + {}^nC_n} = \frac{\sum_{r=0}^n r \cdot {}^nC_r}{2^n}$$

$$= \frac{n \cdot 2^{n-1}}{2^n} = \frac{n}{2}$$

$$\text{Again, } \frac{1}{N} \sum_{i=1}^n f_i x_i^2 = \frac{1}{2^n} \sum_{r=0}^n r^2 {}^nC_r \text{ [where } N = \sum f]$$

$$= \frac{n(n+1)2^{n-2}}{2^n} = \frac{n(n+1)}{4}$$

20. (a) : Here range = r = largest value - smallest value
 $= \text{Max } |x_i - x_j| \text{ (} i \neq j \text{)}$

$$\text{And } S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\text{Now, } (x_i - \bar{x})^2 = \left[x_i - \frac{x_1 + x_2 + \dots + x_n}{n} \right]^2$$

$$= \frac{1}{n^2} [(x_i - x_1) + (x_i - x_2) + \dots + (x_i - x_n)]^2$$

$$= \frac{1}{n^2} [(x_i - x_1) + (x_i - x_2) + \dots + (x_i - x_{i-1}) + (x_i - x_{i+1}) + \dots + (x_i - x_n)]^2$$

$$\Rightarrow (x_i - \bar{x})^2 \leq \frac{1}{n^2} [(n-1)r]^2 \quad (\because |x_i - x_j| \leq r)$$

$$\Rightarrow \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \leq \frac{1}{n^2(n-1)} \sum [(n-1)r]^2$$

(summing up and dividing by $(n-1)$ both sides)

$$= \frac{1}{n^2} \frac{1}{n-1} n(n-1)^2 r^2 = \frac{n-1}{n} r^2 < \frac{n}{n-1} r^2$$

$$\left(\because \forall n > 1, n > \frac{1}{n} \right)$$

Therefore $S^2 < \frac{n}{n-1} \cdot r^2$ or $S < r\sqrt{\frac{n}{n-1}}$

21. (c) : Five numbers can be drawn from 40 numbers in ${}^{40}C_5$ ways, therefore total number of exhaustive cases = ${}^{40}C_5$.

We want that $x_3 = 24$.

\therefore The number of favourable cases are ${}^{23}C_2 \times {}^{16}C_2$

\therefore Required probability = $\frac{{}^{23}C_2 \times {}^{16}C_2}{{}^{40}C_5}$

22. (c) : The total length of the interval = $12 - (-6) = 18$. If graph of $y = x^2 + 2(a+4)x + (3a+40)$ is entirely below x -axis, the value of the discriminant of the above quadratic expression must be negative

$$\begin{aligned} \therefore 4(a+4)^2 - 4 \cdot 1 \cdot (3a+40) &< 0 \\ \Rightarrow a^2 + 5a - 24 < 0 &\Rightarrow (a+8)(a-3) < 0 \\ \Rightarrow -8 < a < 3 \end{aligned}$$

But $a \in [-6, 12]$. $\therefore -6 < a < 3$ for event to happen.

\therefore Length of interval = $3 - (-6) = 9$

Hence, the required probability = $\frac{9}{18} = \frac{1}{2}$

23. (d) : Given, $P(A \cup B) = \frac{5}{6}$, $P(A \cap B) = \frac{1}{3}$, $P(B^c) = \frac{1}{2}$

$$P(A \cup B) = \frac{5}{6} \Rightarrow P(A) + P(B) - P(A \cap B) = \frac{5}{6}$$

$$\Rightarrow P(A) + \frac{1}{2} - \frac{1}{3} = \frac{5}{6} \Rightarrow P(A) = \frac{2}{3}$$

$$\text{Again } P(A) \times P(B) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} = P(A \cap B)$$

$$\therefore P(A) \times P(B) = P(A \cap B)$$

24. (b) : Let E_1 and E_2 are two events that Ram and Shyam will alive 30 years respectively.

$$\text{Given that } P(E_1) = \frac{7}{11} \Rightarrow P(E_1^c) = \frac{4}{11}$$

$$\text{and } P(E_2) = \frac{7}{10} \Rightarrow P(E_2^c) = \frac{3}{10}$$

$$\therefore P(E_1^c \cap E_2^c) = P(E_1^c) P(E_2^c) = \frac{4}{11} \times \frac{3}{10} = \frac{12}{110}$$

25. (a) : Let M be the event that the selected person is a male and S be the event that the selected person is a service holder.

Out of 10 service holder men 1 may be selected in ${}^{10}C_1 = 10$ ways.

So, total number of cases favourable to the event $S \mid M$ is 10 and total number of exhaustive cases is 20.

Hence, the required probability $P(S \mid M) = \frac{10}{20} = \frac{1}{2}$

26. (b) : Let $A(4, -5, 1)$, $B(3, -4, 0)$ and $C(6, -7, 3)$ and $D(7, -8, 4)$

$$AB = \sqrt{1+1+1} = \sqrt{3}, \quad BC = \sqrt{9+9+9} = 3\sqrt{3},$$

$$CD = \sqrt{1+1+1} = \sqrt{3}, \quad AD = \sqrt{9+9+9} = 3\sqrt{3},$$

$$AC = \sqrt{4+4+4} = 2\sqrt{3}, \quad BD = \sqrt{16+16+16} = 4\sqrt{3}$$

Here, $AB = CD$, $BC = AD$ But $AC \neq BD$

Now, midpoint of AC is $(5, -6, 2)$

And midpoint of BD is $(5, -6, 2)$

As midpoint of AC = Midpoint of BD

$\therefore ABCD$ is a parallelogram.

27. (d) : As plane meets co-ordinate axes at

$$A\left(\frac{3}{a}, 0, 0\right), B\left(0, \frac{3}{b}, 0\right) \text{ and } C\left(0, 0, \frac{3}{c}\right)$$

$$\therefore \text{Centroid } G\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

28. (b) : Suppose xy -plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ in the ratio $\lambda : 1$. Then the coordinates

$$\text{of the point of division are } \left(\frac{4\lambda+1}{\lambda+1}, \frac{2\lambda+2}{\lambda+1}, \frac{\lambda+3}{\lambda+1}\right)$$

This point lies on xy -plane.

$$\text{So, } z\text{-coordinate} = 0 \Rightarrow \frac{\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = -3$$

Hence, xy -plane divides the join of point $(1, 2, 3)$ and $(4, 2, 1)$ externally in the ratio $3 : 1$.

29. (b) : Let R divides PQ in ratio $\lambda : 1$

Then coordinates of R given by

$$\left(\frac{5\lambda+3}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-6\lambda-4}{\lambda+1}\right)$$

But given that R has coordinates $(9, 8, -10)$

$$\text{Comparing } x\text{-coordinate, we get } \frac{5\lambda+3}{\lambda+1} = 9$$

$$\Rightarrow 5\lambda+3 = 9\lambda+9 \Rightarrow 4\lambda = -6 \Rightarrow \lambda = \frac{-3}{2}$$

Thus, R divides PQ in the ratio $3 : 2$ externally.

30. (a) : D divides BC in the ratio $AB : AC$ i.e. $3 : 13$

Therefore coordinates of D are

$$\left(\frac{3 \times (-9) + 13 \times 5}{3+13}, \frac{3 \times 6 + 13 \times 3}{3+13}, \frac{3 \times (-3) + 13 \times 2}{3+13}\right)$$

$$= \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

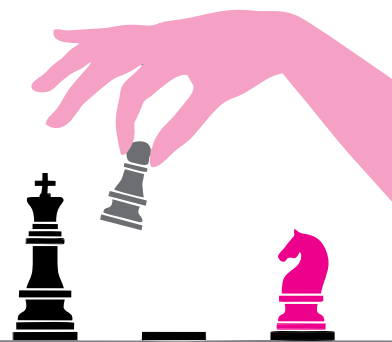


Challenging PROBLEMS



ON

Probability



- Let a and b be distinct, randomly chosen roots of the equation $z^{210} - 1 = 0$. The probability that $\sqrt{2+\sqrt{3}} \leq |a+b|$ is (approx.)
(a) 0.16 (b) 0.24 (c) 0.33 (d) 0.45
- A bag contains 21 red balls and 21 black balls. We remove 2 balls at a time repeatedly and discard them if they are of the same colour, but if they are different then discard the black ball and return the red ball. The probability that this process will terminate with one red ball in the bag is
(a) 0 (b) 0.5 (c) $1/3$ (d) 1
- An exam consists of 3 problems selected randomly from a list of $(2n)$ problems ($n \in \mathbb{N}$, $n > 1$). For a student to pass, he needs to solve correctly atleast 2 of the 3 problems. Knowing that a certain student knows how to solve exactly half of the $(2n)$ problems, the probability that the student will pass the exam is
(a) 0 (b) 0.5 (c) $1/3$ (d) 1
- The probability that in the process of repeatedly flipping a coin, one will get a run of 5 heads before one gets a run of 2 tails is
(a) $\frac{1}{34}$ (b) $\frac{2}{34}$ (c) $\frac{3}{34}$ (d) $\frac{4}{34}$
- A coin of diameter d is thrown randomly on a floor tiled with squares of side s . Two players bet that coin will land on exactly one or more than one square. If the game is to be fair, then $d/s =$
(a) $1 - \frac{1}{\sqrt{2}}$ (b) $2 - \frac{1}{\sqrt{2}}$ (c) $1 - \frac{1}{2\sqrt{2}}$ (d) $2 - \frac{1}{2\sqrt{2}}$
- A coin is tossed 6 times. The probability that 2 heads will turn up in succession somewhere in the sequence is
(a) $\frac{41}{64}$ (b) $\frac{42}{64}$ (c) $\frac{43}{64}$ (d) $\frac{44}{64}$
- 20 chairs are set in a row. 5 people randomly sit on the chairs. The probability that nobody is sitting next to anybody else is
(a) $\frac{91}{323}$ (b) $\frac{81}{331}$ (c) $\frac{72}{314}$ (d) $\frac{64}{315}$
- Select numbers a and b between 0 and 1 independently and at random, and let c be their sum. Let A , B and C be the results when a , b and c respectively are rounded to the nearest integer (Assume that $1/2$ is rounded to 1 and similarly $n + 1/2$ is rounded to $n + 1$ for all positive integers n). Then probability that $A + B = C$ is
(a) $1/2$ (b) $1/3$ (c) $3/4$ (d) $4/5$
- 10 points in the plane are given, with no 3 collinear. 4 distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some 3 of the segments form a triangle whose vertices are among the 10 given points is $\frac{\lambda}{{}^{45}C_4}$ where $\lambda =$
(a) 5040 (b) 4020 (c) 2010 (d) 4030
- Each face of a cube is painted either red or blue, each with probability $1/2$. The colour of each face is determined independently. The probability that the painted cube can be placed on a horizontal surface so that that 4 vertical faces are all of the same colour is
(a) $2/16$ (b) $3/16$ (c) $4/16$ (d) $5/16$
- 40 slips are placed in a hat, each having a number 1, 2, 3, 4, 5, ..., 10 with each number entered on 4 slips. Four slips are drawn at random without replacement from the hat. The probability that two of the slips have a number a and the other two have a number b , $b \neq a$, is $\frac{\lambda \cdot {}^{10}C_2}{{}^{40}C_4}$ where $\lambda =$
(a) 30 (b) 20 (c) 36 (d) 25
- 25 persons are seated at a round table. All choices being equally likely, a team of 3 persons are chosen. The probability that atleast two of the three had been sitting next to each other is
(a) $11/46$ (b) $12/46$ (c) $13/46$ (d) $14/46$

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13. Three players A, B, C take turns to roll a dice in the order ABC, ABC, A... Each player drops out of the game immediately upon throwing a six. The probability that A is the 2nd player to roll a six is

(a) $\frac{300}{1001}$ (b) $\frac{300}{2001}$ (c) $\frac{320}{1001}$ (d) $\frac{320}{2001}$

14. A plant gets two independent genes for flower colour, one from each parent. If the genes are identical, then the flowers are uniformly of that colour. If they are different, then the flowers are striped in those two colours. The genes for the colours pink, crimson and red occur in the population in the ratio of $p : q : r$ where $p + q + r = 1$. A given plant's parent are selected at random. Let A be the event that its flowers are at least partly pink, and let B be the event that its flowers are striped then

(a) $P(A) = 1 - (1 - p)^2$
 (b) $P(B) = 2(pq + pr)$
 (c) $P(A) = 1 - \left(p - \frac{1}{2}\right)^2$ (d) $P(B) = 2(pq + pr + qr)$

15. You enter a betting game with ₹ m and on each spin of a wheel, you bet ₹ 1 at evens on the event E that the result is red. The wheel is not fair. So, $P(E) = p < 1/2$. If you lose all ₹ m , you leave; and if ever have $n \geq m$, you choose to leave immediately. The probability that you leave with nothing is

(a) $\frac{\lambda^n - \lambda^m}{\lambda^n - 1}$ (b) $\frac{\lambda^n - \lambda^m}{\lambda^m - 1}$
 (c) $\frac{\lambda^n - \lambda^m}{\lambda^n + 1}$ (d) $\frac{\lambda^n - \lambda^m}{\lambda^m + 1}$

where $\lambda = \frac{1-p}{p}$

SOLUTIONS

1. (a) : As all the 210 roots of the equation are symmetrically distributed in the complex plane, we can assume $a = 1$ without losing generality. So, the given condition becomes

$$|1 + b|^2 = |1 + \cos\theta + i\sin\theta|^2 = 2 + 2\cos\theta \geq 2 + \sqrt{3}$$

$$\text{i.e., } \cos\theta \geq \frac{\sqrt{3}}{2} \text{ i.e., } |\theta| \leq \frac{\pi}{6}$$

$$\text{As } b \neq 1, \theta \text{ is of the form } \pm \frac{2k\pi}{210}$$

$$\text{where } k \in \left[1, \left\lceil \frac{210}{12} \right\rceil\right] = [1, 17]$$

There are $2 \times 17 = 34$ such angles.

$$\text{So, the required probability} = \frac{34}{209} \approx 0.16$$

2. (d) : As at least one ball is removed during each stage, the process will eventually end with either no ball or one ball. Because red balls are odd in number at start and we remove 2 at a time, the number of red balls at any time is odd. Hence the process will always leave red balls in the bag. So, it ends with exactly 1 red ball. Hence, the required probability is 1.

3. (b) : Define the event E_i as the student solves correctly exactly i of the 3 proposed problems, $i = 0, 1, 2, 3$. The required probability event $E = E_2 \cup E_3$ and $P(E) = P(E_2) + P(E_3)$ as E_2 and E_3 are mutually exclusive events. Because the student knows how to solve exactly half of all the problems. So, $P(E_0) = P(E_3)$ and $P(E_1) = P(E_2)$ and $P(E_0) + P(E_1) + P(E_2) + P(E_3) = 1$. Hence, $P(E) = 1/2$

4. (c) : Let us call 'a successful sequence' a sequence of H and T in which HHHHH appears before TTT does. Each successful sequence must belong to one of the 3 types.

(i) Those beginning with T, followed by a successful sequence that begins with H.

(ii) Those beginning with H, HH, HHH and HHHH, followed by a successful sequence that begins with T.

(iii) The sequence HHHHH.

Let P_H denote the probability of obtaining a successful sequence that begins with H and similarly define P_T .

$$\text{Then, } P_H = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)P_T + \frac{1}{32} \text{ and } P_T = \frac{1}{2}P_H$$

$$\text{Solving, } P_H = \frac{1}{17}, P_T = \frac{1}{34}$$

$$\text{Hence, the required probability} = \frac{3}{34}$$

5. (a) : For the coin to lie entirely on that tile, its centre must fall inside the dotted square of side

$$\text{length, } s - 2 \cdot \frac{d}{2} = s - d$$

This happens with probability

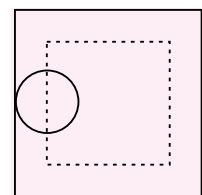
$$= \frac{(s-d)^2}{s^2}$$

$$\text{For the fair game, } \frac{1}{2} = \frac{(s-d)^2}{s^2}. \text{ Hence, } \frac{d}{s} = 1 - \frac{1}{\sqrt{2}}.$$

6. (c) : Let P_n be the probability that no consecutive heads appear in n throws. Then

$$P_n = \frac{1}{2}P_{n-1} + \frac{1}{4}P_{n-2} \text{ with } P_1 = 1, P_2 = \frac{3}{4}. \text{ So, } P_6 = \frac{21}{64}$$

$$\text{Hence, required probability} = 1 - \frac{21}{64} = \frac{43}{64}$$



7. (a) : Total number of cases = no. of solutions in non-negative integers to $(x_1 + x_2 + \dots + x_6 = 15)$
 $= {}^{20}C_5$

And favourable number of cases

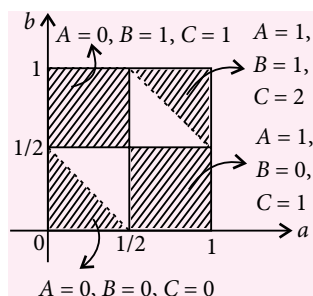
= no. of positive solutions to $y_1 + x_2 + x_3 + x_4 + x_5 + y_6 = 17 = {}^{16}C_5$, where $y_1 = x_1 + 1$, $y_6 = x_6 + 1$

Hence, required probability = $\frac{{}^{16}C_5}{{}^{20}C_5} = \frac{91}{323}$

8. (c) : $A = \begin{cases} 0 & \text{if } a < \frac{1}{2} \\ 1 & \text{if } a \geq \frac{1}{2} \end{cases}$, $B = \begin{cases} 0 & \text{if } b < \frac{1}{2} \\ 1 & \text{if } b \geq \frac{1}{2} \end{cases}$

$C = \begin{cases} 0, & \text{if } a+b < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq a+b < \frac{3}{2} \\ 2, & \text{if } a+b \geq \frac{3}{2} \end{cases}$

$A + B = C$ is only in the shaded region.



So, required probability = $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{3}{4}$

9. (a) : Total line segments = ${}^{10}C_2 = 45$

So, ${}^{45}C_4$ ways of choosing 4 segments.

To count the number of ways of choosing 4 segments such that 3 of them form a triangle. Among the 10 points there are ${}^{10}C_3 = 120$ ways of choosing 3 vertices to form a triangle. There are 45 possible line segments, and we have already chosen three, so there are 42 ways of choosing the 4th segment.

So, total = $120 \times 42 = 5040$ ways.

So, required probability = $\frac{5040}{{}^{45}C_4}$.

10. (d) : Total number of ways = 2^6

Favourable number of ways :

0 red faces \rightarrow no. of cubes possible = 1 with all four vertical faces same colour (blue)

1 red face \rightarrow 6 cubes possible

2 red face \rightarrow 3 cubes possible

3 red face \rightarrow 0 cubes possible

4 red face \rightarrow 3 cubes possible (equivalent of 2 red faces)

5 red face \rightarrow 6 cubes possible

6 red face \rightarrow 1 cube possible

\therefore Required probability = $\frac{1+6+3+0+3+6+1}{2^6} = \frac{5}{16}$

11. (c) : Total ways = ${}^{40}C_4$

Favourable ways = $({}^{10}C_2 \text{ ways of choosing } a \text{ and } b)$

$\times ({}^4C_2 \text{ ways of choosing } a) \times ({}^4C_2 \text{ ways of choosing } b)$

So, required probability

$$= \frac{{}^{10}C_2 \times {}^4C_2 \times {}^4C_2}{{}^{40}C_4} = \frac{36 \times {}^{10}C_2}{{}^{40}C_4}$$

12. (a) : Total ways = ${}^{25}C_3$

Favourable ways = (25×21) ways to choose exactly 2 adjacent persons + (25) ways to choose exactly 3 adjacent persons

So, required probability = $\frac{(25 \times 21) + (25)}{{}^{25}C_3} = \frac{11}{46}$

13. (a) : Consider the sample space consisting of all sequences of length $(3r + 1)$, $(r \geq 0)$ using numbers 1, 2, 3, 4, 5, 6. This represents $(3r + 1)$ rolls of the die. So, total possible outcomes = 6^{3r+1} .

Suppose A is the 2nd player to roll a six on the $(3r + 1)$ th roll. Then his $(r + 1)$ rolls include no six except his last roll. This can occur in 5^r ways. If B was first to roll a six, then his r rolls include any one roll of six = $6^r - 5^r$ ways. In this case, C rolled no six in r attempts = 5^r ways

Hence A is second to B = $5^r \cdot 5^r \cdot (6^r - 5^r)$

Similarly, A is second to C = $5^r \cdot 5^r \cdot (6^r - 5^r)$

Hence, probability $(P_r) = \frac{2 \times 5^r \cdot 5^r \cdot (6^r - 5^r)}{6^{3r+1}}$

So, required probability = $\sum_{r=1}^{\infty} P_r = \frac{300}{1001}$

Note that in calculating the sample space, we have ignored the dropping out case, because it does not affect the respective chances of others to roll a six.

14. (d) : Let $P \rightarrow$ pink, $C \rightarrow$ crimson, $R \rightarrow$ red

$P(PP) = p \cdot p = p^2$, $P(RP) = r \cdot p$

Hence, $P(A) = P(PP \cup PR \cup RP \cup PC \cup CP)$

$$= p^2 + 2pr + 2pq = 1 - (1 - p)^2$$

and $P(B) = P(PC \cup PR \cup RC) = 2(pq + pr + rq)$

15. (a) : Let P_m be the probability that you leave with nothing, then we form a recursive relation for P_m .

$$P_m = P \cdot P_{m+1} + (1 - P)P_{m-1}, 0 < m < n$$

If $m = 0$, then you certainly leave with nothing = $P_0 = 1$

and $m = n$, $P_n = 0$ as you leave before betting

Using these boundary conditions, we have

$$P_m = \frac{\lambda^n - \lambda^m}{\lambda^n - 1}, \text{ where } \lambda = \frac{1-p}{p}$$

Notice that as $n \rightarrow \infty$, $P_m \rightarrow 1$.



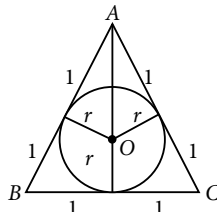
OLYMPIAD CORNER



- In an equilateral triangle ABC (of side length 2) consider the incircle I .
 - Show that for all points P of I ,
 $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 5$.
 - Show that for all points P of I , it is possible to construct a triangle of sides PA, PB, PC , with area $\sqrt{3}/4$.
- The function f is defined on nonnegative integers by: $f(0) = 0$ and $f(2n+1) = 2f(n)$, for $n \geq 0$,
 $f(2n) = 2f(n) + 1$ for $n \geq 1$.
 - Let $g(n) = f(f(n))$. Show that $g(n - g(n)) = 0$ for all $n \geq 0$.
 - For any $n \geq 1$, let $r(n)$ be the least integer r such that $f^r(n) = 0$ (where $f^2(n) = f(f(n))$, $f^3(n) = f(f^2(n))$, etc.). Compute $\liminf_{n \rightarrow \infty} \frac{n}{2^{r(n)}}$.
- Suppose that, for three consecutive years, a certain provincial government reduces what it spends annually on education. The percentage decreases year by year are a, b and c percent, where a, b, c are positive integers in arithmetic progression. Suppose also that the amounts (in dollars) the government spends on education during these same three years are three positive integers in harmonic progression. Find a, b and c .

SOLUTIONS

- Take O as origin, with the x -axis parallel to BC and y -axis along OA . Then the incircle has radius $r = 1/\sqrt{3}$ and A, B, C have coordinates



$$A\left(0, \frac{2}{\sqrt{3}}\right), B\left(-1, \frac{1}{\sqrt{3}}\right), C\left(1, \frac{1}{\sqrt{3}}\right).$$

A point on the incircle has coordinates parameterized by θ , $0 \leq \theta < 2\pi$ given by $P\left(\frac{1}{\sqrt{3}} \cos \theta, \frac{1}{\sqrt{3}} \sin \theta\right)$.

$$(a) \text{ Let } c = \cos \theta, s = \sin \theta. \text{ Then } AP^2 + BP^2 + CP^2 = \left[\frac{1}{3}c^2 + \frac{1}{3}(2-s)^2\right] + \left[\frac{1}{3}(c+\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right] + \left[\frac{1}{3}(c-\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right]$$

$$= \frac{1}{3} [c^2 + 4 - 4s^2 + s^2 + c^2 + 2\sqrt{3}c + 3 + s^2 + 2s + 1 + c^2 - 2\sqrt{3}c + 3 + s^2 + 2s + 1]$$

$$= \frac{1}{3} [3(c^2 + s^2) + 12] = 5 \quad [\because c^2 + s^2 = 1]$$

$$(b) \text{ Now, set } x = AP = \frac{1}{\sqrt{3}} \sqrt{5-4s},$$

$$y = BP = \frac{1}{\sqrt{3}} \sqrt{5+2s+2\sqrt{3}c}, \text{ and}$$

$$z = CP = \frac{1}{\sqrt{3}} \sqrt{5+2s-2\sqrt{3}c}.$$

By reflection and rotational geometry the distances AP, BP, CP will be a permutation of those obtained when $\pi/6 \leq \theta \leq \pi/2$, so that $1/\sqrt{3} \leq x \leq 1$. Also

$$y = \frac{1}{\sqrt{3}} \sqrt{5+4\left(\frac{1}{2}s - \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}} \sqrt{5+4\cos\left(\theta - \frac{\pi}{6}\right)}$$

so that $\sqrt{7}/\sqrt{3} \leq y \leq \sqrt{3}$ and

$$z = \frac{1}{\sqrt{3}} \sqrt{5+4\left(\frac{1}{2}s - \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}} \sqrt{5+4\sin\left(\theta - \frac{\pi}{3}\right)}$$

so that $\sqrt{5}/\sqrt{3} \leq z \leq \sqrt{7}/\sqrt{3}$. Thus

$$(x+y)_{\min} = \frac{1+\sqrt{7}}{\sqrt{3}} > z_{\max} = \frac{\sqrt{7}}{\sqrt{3}}$$

$$\text{and } (y+z)_{\min} = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{3}} > x_{\max} = 1$$

$$\text{and } (z+x)_{\min} = \frac{1+\sqrt{5}}{\sqrt{3}} > y_{\max} = \sqrt{3},$$

and x, y, z (i.e., AP, BP, CP) can form the sides of a triangle.

From Heron's formula the area, F , of this triangle is given by

$$\begin{aligned} F^2 &= \frac{1}{2}(x+y+z) \cdot \frac{1}{2}(-x+y+z) \cdot \frac{1}{2}(x-y+z) \\ &\quad \cdot \frac{1}{2}(x+y-z) \\ &= \frac{1}{16} [(y+z)^2 - x^2][x^2 - (y-z)^2] \\ &= \frac{1}{16} \left[\frac{1}{3}(5+2s+2\sqrt{3}c+5+2s-2\sqrt{3}c-5+4s) \right. \\ &\quad \left. + 2 \cdot \frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(c+2s-2\sqrt{3}c)^{1/2} \right] \times \\ &\quad \left[\frac{1}{3}(5-4s-5-2s-2\sqrt{3}c-5-2s+2\sqrt{3}c) \right. \\ &\quad \left. + 2 \cdot \frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(5+2s-2\sqrt{3}c)^{1/2} \right] \\ &= \frac{1}{48} \left[(5+8s)+2\sqrt{(5+2s)^2-12c^2} \right] \\ &\quad \left[-(5+8s)+2\sqrt{(5+2s)^2-12c^2} \right] \\ &= \frac{1}{48} [100+80s+16s^2-48c^2-25-80s-64s^2] \\ &= \frac{1}{48} [75-48] = \frac{9}{16}, \quad \text{since } c^2+s^2=1 \end{aligned}$$

Thus $F = 3/4$.

2. (a) Most of the solution can be worked out suitably in binary notation. To prepare for this, we first prove the following lemma.

Lemma : Let $L(n) = 2^{\lfloor \log_2(2n) \rfloor}$, $n \geq 1$,

where $[x]$ denotes the greatest integer $\leq x$. Then $f(n) = L(n) - n - 1$.

Proof : Since $1 + \lfloor \log_2(2n) \rfloor = \lfloor 1 + \log_2(2n) \rfloor = \lfloor \log_2(4n) \rfloor$, we have $2L(n) = L(2n)$. Furthermore, we claim that $L(2n+1) = L(2n)$.

Suppose $\lfloor \log_2(2n) \rfloor < \lfloor \log_2(2n+1) \rfloor$. Then letting $\lfloor \log_2(2n) \rfloor = k$, we have $\log_2(2n) < k+1 \leq \log_2(2n+1)$ which implies $2n < 2^{k+1} \leq 2n+1$ or $n < 2^k \leq n+1/2$, clearly an impossibility.

Since $f(1) = 0$ and $L(1) = 2$, $f(n) = L(n) - n - 1$ holds for $n = 1$. Assume the formula holds up to $2n-1$ for some $n \geq 1$. Then

$$\begin{aligned} f(2n+1) &= 2f(n) = 2L(n) - 2n - 2 \\ &= L(2n) - 2n - 2 = L(2n+1) - (2n+1) - 1 \\ \text{and } f(2n) &= 2f(n) + 1 = L(2n) - 2n - 1. \end{aligned}$$

This completes the proof of the lemma.

Since $g(0) = f(f(0)) = f(0) = 0$, $g(0 - g(0)) = 0$. Assume henceforth that $n \geq 1$. The binary representation of n can be written in the form

$$n = \underbrace{1 \dots 1}_s \underbrace{0 \dots 0}_t \underbrace{1^* \dots^*}_u,$$

where each $*$ is either 0 or 1, and $s \geq 1$, $t, u \geq 0$. Since $L(n) - 1$ is just a string of 1's in binary notation, and since $n + f(n) = L(n) - 1$ by the lemma, we see that f acts on n by interchanging all 1's and 0's in the binary expansion of n . In other words,

$$f(n) = \underbrace{1 \dots 1}_t \underbrace{0^* \dots^*}_u$$

where each $*$ is $1 - *$. Thus,

$$g(n) = f(f(n)) = \underbrace{1^* \dots^*}_u;$$

i.e., $g(n)$ is the original final block of u digits in n . As a result,

$$n - g(n) = \underbrace{1 \dots 1}_s \underbrace{0 \dots 0}_{t+u}$$

Since the last expression contains no final block of digits after the initial blocks of 1's and 0's, we have $g(n - g(n)) = 0$ for all $n \geq 1$.

(b) The answer is $2/3$. To see this, it is necessary to examine more closely the blocks of binary digits of n . Since

$$n = \begin{cases} \underbrace{1 \dots 1}_{s_1} \underbrace{0 \dots 0}_{s_2} \dots \underbrace{1 \dots 1}_{s_m} & \text{if } m \text{ is odd,} \\ \underbrace{1 \dots 1}_{s_1} \underbrace{0 \dots 0}_{s_2} \dots \underbrace{0 \dots 0}_{s_m} & \text{if } m \text{ is even,} \end{cases}$$

and since f always wipes out the leading block of 1's, it is clear that $r(n) = m$. Since n has at least $r(n)$ digits and $2^{r(n)}$ has $1 + r(n)$ digits, $n/2^{r(n)}$ will exceed 1 unless n has exactly $r(n)$ digits; i.e., unless n is in the sequence (in base 2) 1, 10, 101, 1010, 10101, So the limit inferior of $n/2^{r(n)}$ is the limit of the sequence (in base 2)

$$\frac{1}{10}, \frac{10}{10^2}, \frac{101}{10^3}, \dots,$$

which is (in base 10)

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots = \frac{2}{3}.$$

3. The government spending in each year is given by:

Year Spending

$$0 \quad N$$

$$1 \quad N_1 = N \left(1 - \frac{a}{100} \right)$$

$$2 \quad N_2 = N \left(1 - \frac{a}{100} \right) \left(1 - \frac{b}{100} \right)$$

$$3 \quad N_3 = N \left(1 - \frac{a}{100} \right) \left(1 - \frac{b}{100} \right) \left(1 - \frac{c}{100} \right)$$

Now a, b, c in arithmetic progression

$$\Rightarrow c = 2b - a,$$

and N_1, N_2, N_3 in harmonic progression

$$\Rightarrow \frac{1}{N_2} = \frac{1}{2} \left(\frac{1}{N_1} + \frac{1}{N_3} \right).$$

Cancelling the factor $N \left(1 - \frac{a}{100} \right)$ in the denominators of the second equation yields

$$\begin{aligned} \frac{2}{1 - \frac{b}{100}} &= 1 + \frac{1}{\left(1 - \frac{b}{100} \right) \left(1 - \frac{c}{100} \right)} \\ &= 1 + \frac{1}{\left(1 - \frac{b}{100} \right) \left(1 - \frac{2b-a}{100} \right)}, \end{aligned}$$

which implies

$$2 \left(1 - \frac{2b-a}{100} \right) = \left(1 - \frac{b}{100} \right) \left(1 - \frac{2b-a}{100} \right) + 1,$$

$$200(100 + a - 2b) = (100 - b)(100 + a - 2b) + 10000,$$

and thus $(100 + a - 2b)(100 + b) = 10000$.

Since $1 \leq a, b \leq 99$ we have $101 \leq 100 + b \leq 199$, and the only (positive integer) factorization of 10000 with one factor between 101 and 199 is 80×125 . Hence $100 + b = 125$ and $100 + a - 2b = 80$, so $b = 25, a = 30, c = 20$.

It may be of interest to note that a unique solution also exists in the case when N_1, N_2 and N_3 (the

three annual amounts spent) are in arithmetic progression. Since in this case $N_2 - N_1 = N_3 - N_2$

we have, again cancelling a factor of $N \left(1 - \frac{a}{100} \right)$ throughout,

$$\begin{aligned} \frac{b}{100} &= 1 - \frac{b}{100} - 1 = \left(1 - \frac{b}{100} \right) \left(1 - \frac{c}{100} - 1 \right) \\ &= \left(1 - \frac{b}{100} \right) \left(\frac{2b-a}{100} \right), \end{aligned}$$

which implies $100b = (100 - b)(2b - a)$

and hence $2b^2 - (100 + a)b + 100a = 0$.

For b to be rational we must have

$$(100 + a)^2 - 4 \cdot 2 \cdot 100a = a^2 - 600a + 10000 = k^2$$

for some integer k , i.e.,

$$(300 - a)^2 - 80000 = k^2. \quad \dots(i)$$

As $1 \leq a \leq 99$ and $300 - a \geq \sqrt{80000}$, a must be at most 17. Trial gives the only suitable value of a to be 15, whence $2b^2 - 115b + 1500 = 0$ and so $b = 20$ (being an integer) and $c = 25$. [Here is an alternative way to get $a = 15$. Since $a \geq 1$, from (i) we get $k^2 \leq 299^2 - 80000 = 9401$, so $k \leq 96$. Rewriting (i) as

$$(300 - a - k)(300 - a + k) = 80000,$$

we see that we need to factor 80000 into two factors which differ by at most 192. The only such factorization is $80000 = 250 \times 320$, which yields $k = 35$ and $a = 15$.]

Exam Update

JEE (Advanced) 2018

The Joint Entrance Examination (Advanced) 2018 will be conducted by the IITs. The performance of a candidate in this examination will form the basis for admission to the Bachelor's, Integrated Master's and Dual Degree programs (entry at the 10+2 level) in all the IITs. The decisions of the JAB 2018 will be final in all matters related to JEE (Advanced) 2018 and admission to IITs.

Candidates should be among the top 2,24,000 (including all categories) by scoring positive marks in Paper-1 of JEE (Main)-2018.

Examination will be held on May 20, 2018. The entire JEE (Advanced) 2018 Examination will be conducted in fully computer based test mode. The exam consists of two papers, Paper 1 and Paper 2, each of three hours' duration, and will be held in two sessions. Both the papers are compulsory.

For more information visit www.jeeadv.ac.in

MATH archives



Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The readers' & comments and suggestions regarding the problems and solutions offered are always welcome.

- Two finite sets have m and n elements. The total number of subsets of first set is 56 more than the total number of subsets of the second set. Then $(m, n) =$
(a) (7, 6) (b) (6, 3) (c) (5, 1) (d) (8, 7)
- If $\lim_{x \rightarrow e^3} \frac{[(\log_e x) - 3]^n}{\log_e \left\{ \cos^m [(\log_e x) - 3] \right\}} = -1$ ($n, m \in N$)
then n/m is equal to
(a) 3 (b) 4 (c) 9 (d) 1
- The roots of the equation $x^2 + 2(a - 3)x + 9$ lies between -6 and 1 then $[a] = \underline{\hspace{1cm}}$, where $[.]$ denotes greatest integer function, then
(a) 3 (b) 6 (c) 12 (d) 19
- The total number of 6-digit numbers that are divisible by 5, is
(a) 180000 (b) 540000
(c) 5×10^5 (d) 200000
- Statement-I: $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.
Statement -II: $\sim(p \leftrightarrow \sim q)$ is a tautology.
(a) Both (I) and (II) are true and (II) is correct explanation of (I).
(b) Both (I) and (II) are true and (II) is not correct explanation of (I).
(c) (I) is true but (II) is false.
(d) (I) is false but (II) is true.
- If α is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$ ($b < c$), then the value of $\tan^{-1} \alpha + \tan^{-1} \left(\frac{1}{\alpha} \right)$ is equal to
(a) $\pi/2$ (b) $-\pi/2$
(c) 0 (d) does not exist
- The letters of the word "ORIENTAL" are arranged in all possible ways, the chance that the consonants and vowels occur alternatively is
(a) $\frac{1}{70}$ (b) $\frac{2}{35}$
(c) $\frac{1}{35}$ (d) $\frac{4}{35}$
- Two parabolas have the same focus at (3, 2) and their directrices are the x -axis and y -axis respectively, then the slope of their common chord is
(a) -1 (b) $-1/2$
(c) $-\frac{\sqrt{3}}{2}$ (d) 1
- If $2ax + 3by + 4c = 0$ is an equation of the line joining the extremities of a pair of semi-conjugate diameter of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, then $\frac{a^2 + b^2}{c^2} =$
(a) $4/9$ (b) 1
(c) $8/9$ (d) $16/9$
- If $f(x) = 4 - (6 - x)^{2/3}$ in $[5, 7]$, then on $f(x)$
(a) Lagrange's theorem is applicable
(b) Rolle's theorem is applicable
(c) Lagrange's and Rolle's theorem are applicable
(d) Lagrange's and Rolle's theorem are not applicable

By : Prof. Shyam Bhushan, Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

SOLUTIONS

1. (b) : $2^m = 2^n + 56$
 $\Rightarrow (m, n) = (6, 3)$ satisfies the equation.
2. (d) : Let $\ln x - 3 = t$
 $\therefore \lim_{t \rightarrow 0} \frac{t^n}{\ln(\cos^m t)} = -1 \quad \left(\frac{0}{0} \text{ form} \right)$
 $\Rightarrow \lim_{t \rightarrow 0} \frac{nt^{n-1}}{-m \tan t} = -1$
 $\Rightarrow n - 1 = 1 \text{ and } -\frac{n}{m} = -1$
 $\Rightarrow n = m = 2.$
3. (b) : $\Delta \geq 0, f(-6) > 0, f(1) > 0, -6 < \frac{\alpha + \beta}{2} < 1$
 $\Rightarrow [a] = 6$
4. (a) : Required number of ways
 $= 9 \times 10 \times 10 \times 10 \times 10 \times 2 = 18000$

5. (c) :

p	q	$p \leftrightarrow q$	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	T	T	F	T
6. (b) : $f(0) > 0, f(-1) < 0 \Rightarrow \alpha$ lies between -1 and 0 ,
 $\tan^{-1}(1/\alpha) = -\pi + \cot^{-1}\alpha$
7. (c) : $n(S) = 8!, n(E) = 2(4!)(4!)$
8. (d) : Let the focus be F and the parabolas intersect at P and Q . From P draw perpendiculars on the x -axis and y -axis at A and B respectively, then
 $PA = PF = PB \Rightarrow P, Q$ lies on the line $y = x$
9. (c) : $2ax + 3by + 4c = 0$ passes through $(3\cos\theta, 2\sin\theta)$ and $(-3\sin\theta, 2\cos\theta)$
10. (d) : $f(5) \neq f(7)$ and it is not differentiable at $x = 6$



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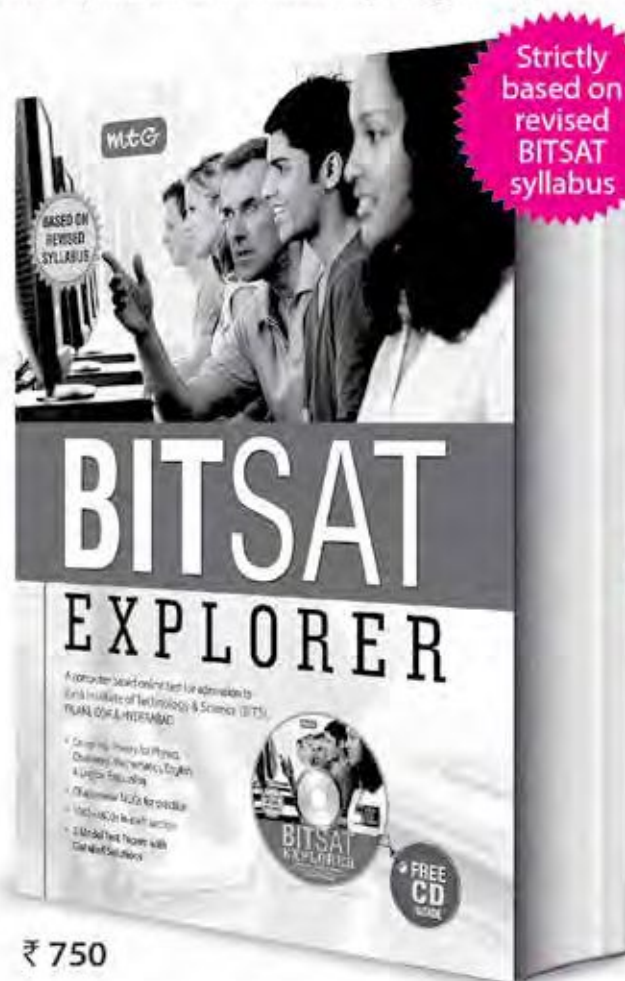


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MATHS MUSING

SOLUTION SET-178

1. (d) : The given equation is

$$(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$$

For this equation to have real roots, $D \geq 0$. Thus,

$$\cos^2 p - 4 \sin p (\cos p - 1) \geq 0$$

$$\text{or } \cos^2 p - 4 \sin p \cos p + 4 \sin^2 p + 4 \sin p - 4 \sin^2 p \geq 0$$

$$\text{or } (\cos p - 2 \sin p)^2 + 4 \sin p (1 - \sin p) \geq 0$$

For every real value of p , we have

$$(\cos p - 2 \sin p)^2 \geq 0 \text{ and } \sin p (1 - \sin p) \geq 0$$

$$\therefore D \geq 0, \forall p \in (0, \pi)$$

2. (a) : Since angles of same segment are equal

$$\therefore \angle BED = \angle BAD = A/2$$

$$\angle BEF = \angle BCF = C/2$$

Now, $\angle DEF = \angle BEF + \angle BED$

$$= \frac{A+C}{2} = 90^\circ - \frac{B}{2}$$

$$\text{Similarly, } \angle DFE = 90^\circ - \frac{C}{2},$$

$$\angle EDF = 90^\circ - \frac{A}{2}$$

$$\text{Area of } \triangle DEF = \frac{1}{2} (DE)(DF) \sin \left(90^\circ - \frac{A}{2} \right)$$

$$= \frac{1}{2} (DE)(DF) \cos \frac{A}{2}$$

Circumradius of $\triangle ABC$ and $\triangle DEF$ are equal.

$$\frac{DE}{\sin(90^\circ - C/2)} = \frac{DF}{\sin(90^\circ - B/2)} = \frac{EF}{\sin(90^\circ - A/2)} = 2R$$

$$DE = 2R \cos(C/2), DF = 2R \cos(B/2), EF = 2R \cos(A/2)$$

$$\text{Area of } \triangle DEF = \frac{1}{2} [2R \cos(C/2) [2R \cos(B/2)] \cos(A/2)]$$

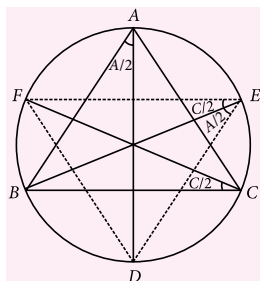
$$= 2R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 2R^2 \sqrt{\frac{s(s-a)}{bc} \cdot \frac{s(s-b)}{ca} \cdot \frac{s(s-c)}{ba}} = \frac{2R^2 s \Delta}{abc}$$

$$\frac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} = \frac{\frac{2R^2 s \Delta}{abc}}{\Delta} = \frac{2R^2 s}{4R \Delta} = \frac{Rs}{2\Delta} = \frac{R}{2r}$$

3. (d) : Let $s = \sin \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} + \frac{1}{3} \sin \frac{3\pi}{3} + \dots \infty$

$$c = \cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} + \frac{1}{3} \cos \frac{3\pi}{3} + \dots \infty$$



$$c + is = \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) + \frac{1}{2} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) + \frac{1}{3} \left(\cos \frac{3\pi}{3} + i \sin \frac{3\pi}{3} \right) + \dots \infty$$

$$= e^{i\pi/3} + \frac{1}{2} \left(e^{i\pi/3} \right)^2 + \frac{1}{3} \left(e^{i\pi/3} \right)^3 + \dots \infty$$

$$= -\log \left(1 - e^{i\pi/3} \right) \left\{ \because x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty = -\log(1-x) \right\}$$

$$= -\log \left(1 - \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right) = -\log \left(\left(1 - \frac{1}{2} \right) - i \frac{\sqrt{3}}{2} \right)$$

$$= -\log \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = - \left(\log \sqrt{\frac{1}{4} + \frac{3}{4}} + i \text{Arg} \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \right)$$

$$(\because \log z = \log |z| + i \text{Arg } z)$$

$$= - \left(\log 1 + i \tan^{-1}(-\sqrt{3}) \right) = i \tan^{-1} \sqrt{3} \Rightarrow s = \frac{\pi}{3}$$

4. (c) : We have $x = t^2 + t + 1$, and $y = t^2 - t + 1$

$$\text{Eliminating } t, \text{ we get, } \frac{x+y}{2} = t^2 + 1, \frac{x-y}{2} = t$$

$$2(x+y) = (x-y)^2 + 4$$

Since the second-degree terms form a perfect square, it represents a parabola (also, $\Delta \neq 0$).

5. (b) : Integers greater than 6000 may be 4 digit or 5 digit numbers.

$$\text{Number of 4 digit numbers} = 3 \times 4 \times 3 \times 2 = 72$$

$$(\because 1^{\text{st}} \text{ digit can be 6, 7 or 8})$$

$$\text{Number of 5 digit numbers} = 5! = 120$$

$$\therefore \text{Total numbers} = 72 + 120 = 192$$

6. (b) : We have $F(x) = \int_0^{x^2} f(\sqrt{t}) dt \Rightarrow F(0) = 0$

$$F'(x) = 2x f(x) = f'(x)$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int 2x dx \Rightarrow \log_e f(x) = x^2 + c$$

$$\Rightarrow f(x) = e^{x^2+c} \Rightarrow f(x) = e^{x^2} \quad (\because f(0) = 1)$$

$$\Rightarrow F(x) = \int_0^{x^2} e^t dt \Rightarrow F(x) = e^{x^2} - 1$$

$$\Rightarrow F(2) = e^4 - 1$$

7. (c)

8. (c)

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \Rightarrow \left[\frac{dy}{dx} \right]_{(t, f(t))} = f'(t)$$

Equation of tangent at $(t, f(t))$ is $y - f(t) = f'(t)(x - t)$
It passes through $(t + 2, 0)$ then $-f(t) = f'(t)(t + 2 - t)$

$$\Rightarrow \frac{f'(t)}{f(t)} = -\frac{1}{2}$$

On integrating we get, $\log_e(f(t)) = -\frac{1}{2}t + c$

$$\Rightarrow f(x) = e^{-\frac{1}{2}x+c}$$

Since curve passes through $(0, 2)$ we have $2 = e^c$

$$\Rightarrow c = \log 2 \Rightarrow y = f(x) = 2e^{-\frac{1}{2}x}$$

$$\therefore \lim_{x \rightarrow \infty} \left(2e^{-\frac{1}{2}x} \right) = 0$$

$$f'(x) = -e^{-\frac{1}{2}x}, f'(0) = -1$$

\therefore Slope of normal at $x = 0$ is 1.

\therefore Equation of normal at $(0, 2)$ is $y - 2 = 1(x - 0)$

or $x - y + 2 = 0$

9. (1) : Equation of normal at the point $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x)$$

$$\text{Let } m = \frac{dy}{dx} \Rightarrow X + mY - (x + my) = 0 \quad \dots(i)$$

Perpendicular distance from the origin to line (i) is

$$\frac{|x + my|}{\sqrt{1 + m^2}} = |y| \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This is homogeneous equation \therefore Let $y = zx$,

$$\Rightarrow \frac{dy}{dx} = z + x \frac{dz}{dx} \Rightarrow \frac{2z}{1 + z^2} dz = -\frac{dx}{x}$$

$$\text{Integrating } \int \frac{2z}{1 + z^2} dz = -\int \frac{dx}{x}$$

$$\Rightarrow \log(1 + z^2) = -\log x + c \Rightarrow (x^2 + y^2) = x \cdot e^c$$

This curve passes through $(1, 1) \Rightarrow 1 + 1 = 1 \cdot e^c \Rightarrow e^c = 2$

The required equation of the curve is $x^2 + y^2 = 2x$

$$\mathbf{10. P.} \quad \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$$

$$= \frac{1}{x^n} \left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) - 1 \right) \left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \right)$$

$$= \frac{1}{x^n} \left[\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \left(-x - x^2 - \frac{x^3}{3!} - \dots \right) \right]$$

$$= \frac{-1}{x^{n-3}} \left[\left(-\frac{1}{2!} + \frac{x^2}{4!} - \dots \right) \left(1 + x + \frac{x^2}{3!} + \dots \right) \right]$$

\therefore For $\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ to exist we must have
 $n - 3 = 0 \Rightarrow n = 3$.

$$\mathbf{Q.} \quad \lim_{x \rightarrow 0} \frac{(1+x)^{1/2} - (1+x)^{1/3}}{x}$$

$$= \lim_{x \rightarrow 0} \left[\frac{1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2}x^2 + \dots}{x} \right]$$

$$- \lim_{x \rightarrow 0} \left[\frac{1 + \frac{1}{3}x + \frac{(1/3)(-2/3)}{2}x^2 + \dots}{x} \right]$$

$$= \lim_{x \rightarrow 0} \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \text{terms containing } x \right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

R. We have, $4 \lim_{x \rightarrow \frac{3}{2}} (x - [x])$

$$\text{L.H.L.} = \lim_{h \rightarrow 0} \left(\frac{3}{2} - h - \left[\frac{3}{2} - h \right] \right) = \lim_{h \rightarrow 0} \left(\frac{3}{2} - h - 1 \right) = \frac{1}{2}$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} \left(\frac{3}{2} + h - \left[\frac{3}{2} + h \right] \right) = \lim_{h \rightarrow 0} \left(\frac{3}{2} + h - 1 \right) = \frac{1}{2}$$

$$\text{L.H.L.} = \text{R.H.L.} = \frac{1}{2}$$

$$\mathbf{S.} \quad \lim_{x \rightarrow 0} [x] \left(\frac{e^{1/x} - 1}{e^{1/x} + 1} \right)$$

$$\text{L.H.L.} = \lim_{h \rightarrow 0} [0 - h] \left(\frac{e^{\frac{1}{0-h}} - 1}{e^{\frac{1}{0-h}} + 1} \right)$$

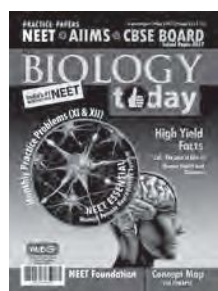
$$\lim_{h \rightarrow 0} (-1) \left(\frac{e^{-\frac{1}{h}} - 1}{e^{-\frac{1}{h}} + 1} \right) = (-1) \times (-1) = 1$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} [0 + h] \left(\frac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1} \right) = 0$$

Limit does not exist.



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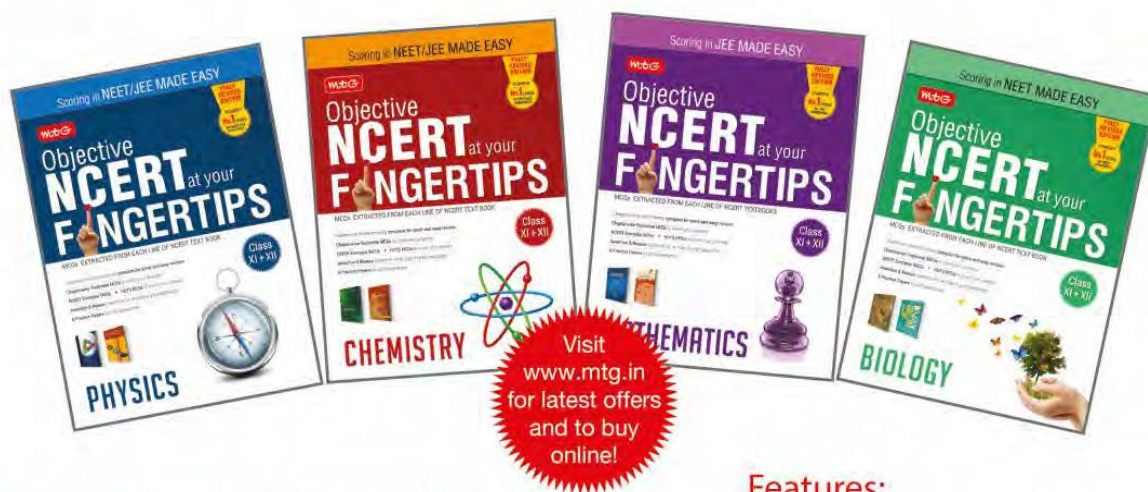
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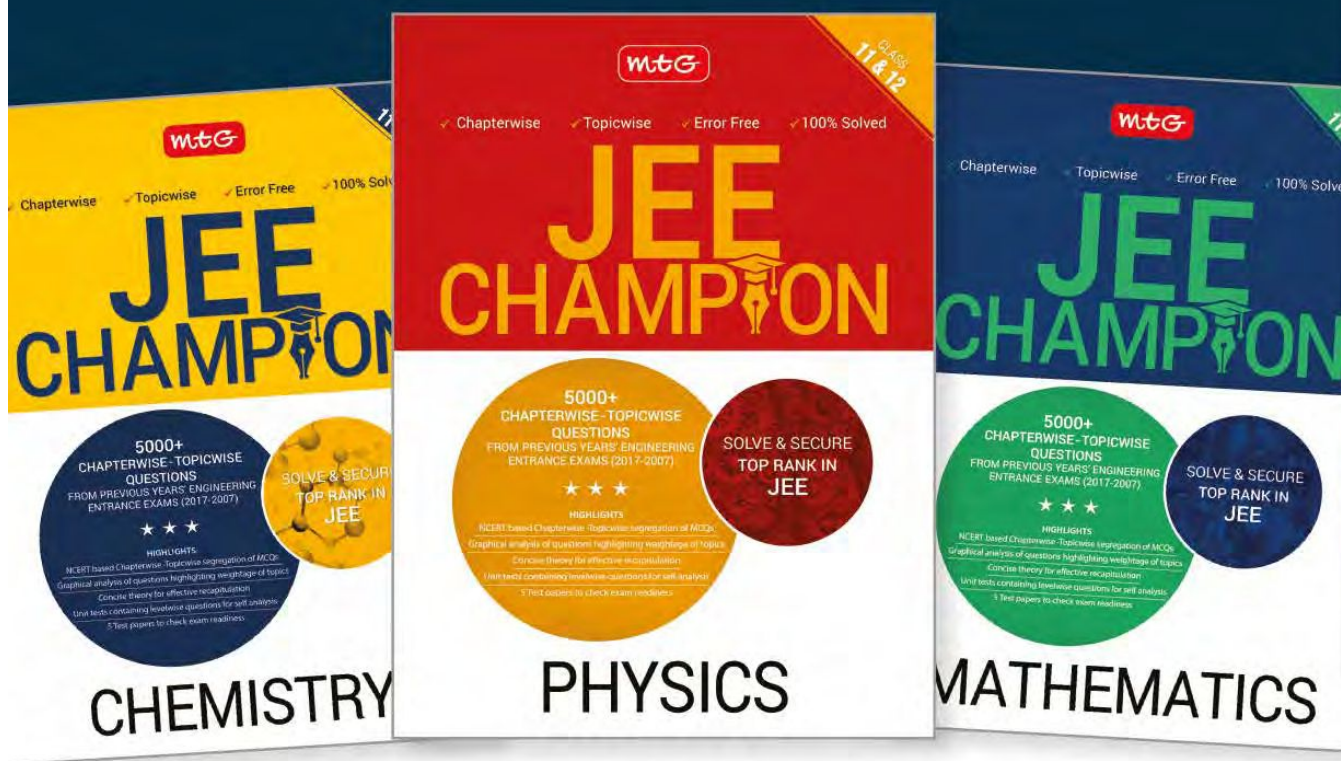
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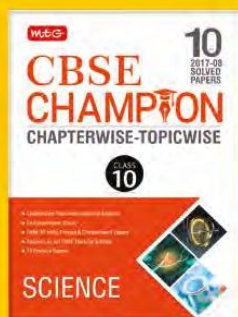
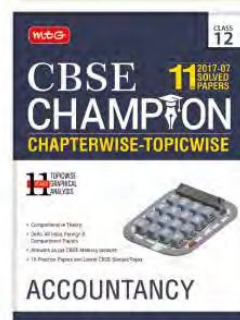
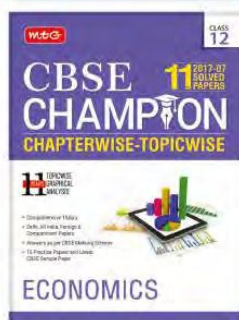
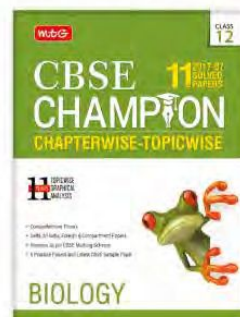
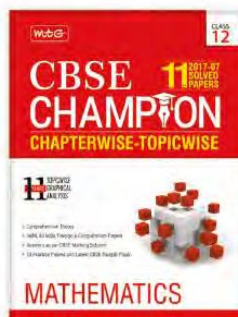
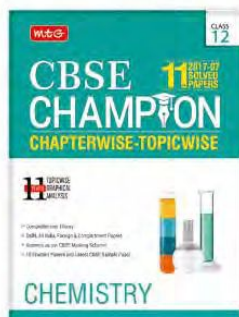
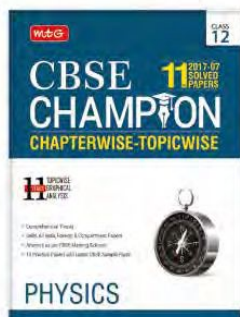
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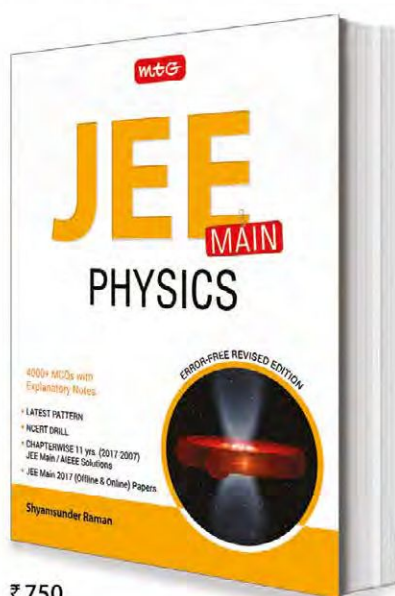


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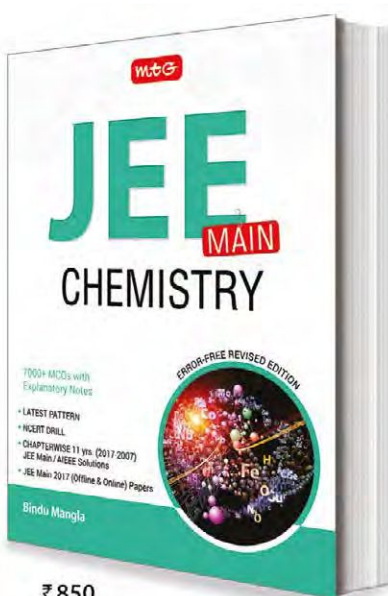
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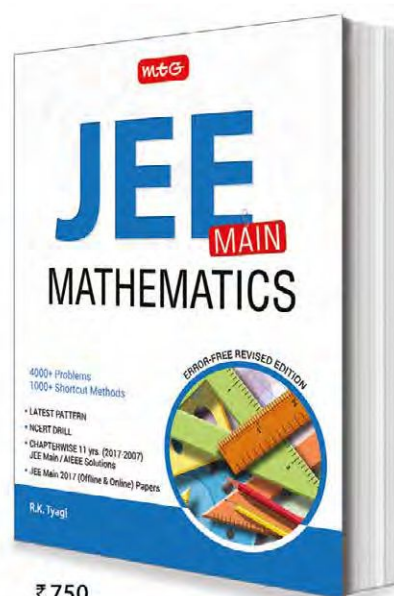
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